PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Karman Street of Vortices in a Channel of Finite Breadth

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VII. *The Kármán Street of Vortices in a Channel of Finite Breadth.*

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1. The following investigations deal with a Kármán street of vortices, or unsymmetrical double row, in a channel of finite width. A discussion on the symmetrical double row has also been incorporated. For the sake of convenience the work has been divided into two distinct parts—(I) Systems of Line Vortices in a Channel of Finite Breadth, and (II) The Kármán Drag Formula for a Channel of Finite Breadth. Part I has itself been divided into two sections headed Problem I and Problem II, and these sections deal with the stream lines and stability of the unsymmetrical and symmetrical double row respectively. Part II discusses Problem I from the point of view of hydrodynamics and contains a section dealing with the hydrodynamical significance of the stability investigations. Summaries of the investigations have been kept distinct and can be found in paragraphs (1.5) and (1.3) of Part I and Part II respectively.

Part I.—Systems of Line Vortices in a Channel of Finite Breadth.

1. *Introduction.*

1.1. The form of the stream lines associated with double rows of line vortices, and the stability of such systems, have been discussed by von Kármán.* Von Kármán, however, published only the outlines of his investigations and more details are given by Lamb.† Their papers deal with two configurations of vortices—firstly, there is the *unsymmetrical* double row, that is, two infinite parallel rows of line vortices, of equal strength and opposite sign, arranged so that each vortex of one row is opposite the centre of the interval between two consecutive vortices on the other row; and secondly, there is the *symmetrical* double row, in which the above system is modified so that each vortex of one row is exactly opposite a vortex on the other row.

1.2. The results obtained by von Kármán can be stated briefly as follows: If κ

* 'Phys. Z.,' vol. 13, p. 53 (1912), and 'Gött. Nachr.,' p. 547 (1912).

† 'Hydrodynamics,' p. 208 (1924).

be the strength of the vortices, $2a$ the distance between the two rows, $2b$ the distance between consecutive vortices, then

I. In the *unsymmetrical* double row we get:

(α) The ω function used by VON KÁRMÁN is equivalent to

$$-\frac{i\kappa}{2\pi}\log\frac{\sin\pi(z-ia)/2b}{\cos\pi(z+ia)/2b}.$$

(β) The system is stable when and only when $\dfrac{a}{b}=\dfrac{1}{\pi}\cosh^{-1}\sqrt{2}=0\cdot281.$

(γ) All the vortices move forward with a velocity $\dfrac{\kappa}{4b}\tanh\pi\dfrac{a}{b}$.

(δ) The stream lines relative to the moving vortices are such, that each vortex is completely surrounded by some stream lines, so that the liquid within these stream lines accompanies the vortex in its career, and the remaining stream lines form a sinuous channel between these portions of the fluid (as in fig. 4 (g), if the barriers are imagined to be at infinity).

II. In the *symmetrical* double row we get:

(α) All the vortices move forward with a velocity $\dfrac{\kappa}{4b}\coth\pi\dfrac{a}{b}$.

(β) The system is unstable.

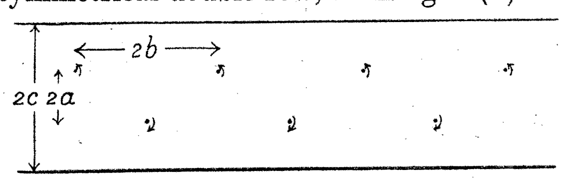
1.3. The original VON KÁRMÁN investigations deal with vortices in an infinite sea of liquid, and this is equivalent to neglecting the effect of the barriers which, in practice, must always be present. The object of this paper is to investigate the stream lines and stability of the above systems of vortices when they are placed between two plane barriers, which, of course, are parallel to the rows. This type of motion is of importance as it suggests the two-dimensional turbulent flow between parallel plane barriers.

1.4. There are two distinct problems to be considered:

I. The bounded unsymmetrical double row, as in fig. 1 (A).

II. The bounded symmetrical double row, as in fig. 1 (B).

(A)



(B)

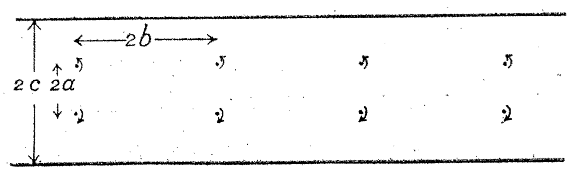


FIG. 1.

We define $2c$ to be the distance between the barriers, and κ, a and b, as in paragraph 1.2. In the ensuing work it will be necessary to introduce certain ϑ functions and we here define our notation. The ϑ function notation of TANNERY and MOLK, 'Fonctions Elliptiques' will be adhered to throughout. For problem I, the periods of these functions, $2\omega_1$ and $2\omega_3$, will be found to be $2b$ and $i4c$ respectively, so that

$$\tau \equiv 2\omega_3/2\omega_1 = i2c/b = it,$$

where we introduce the definition $t = 2c/b$, and where

$$q = \exp(i\pi\tau) = \exp(-\pi t), \qquad q_1 = \exp(-i\pi/\tau) = \exp(-\pi/t).$$

For problem II, $2\omega_1$ and $2\omega_3$ will be seen to be $2b$ and $i2c$ respectively, thus giving

$$\tau \equiv 2\omega_3/2\omega_1 = ic/b = it,$$
$$q = \exp(-\pi t), \qquad q_1 = \exp(-\pi/t),$$

where $t = c/b$. Let us also introduce the definitions

$$d = a/2b, \qquad \mu = \pi b/2c, \qquad \nu = \pi a/2c,$$
$$Z = X + iY = (x + iy)/2b = z/2b.$$

1.5. *Main Results of the Present Investigations.*

Problem I.

(α) The ω function is $-\dfrac{i\kappa}{2\pi}\log\dfrac{\vartheta_1(Z-id)\,\vartheta_3(Z-id)}{\vartheta_2(Z+id)\,\vartheta_4(Z+id)}$, where $\tau = i\,\dfrac{2c}{b}$.

(β) All the vortices move forward with a velocity $\dfrac{i\kappa}{4\pi b}\left[\dfrac{\vartheta'_2(2id)}{\vartheta_2(2id)} + \dfrac{\vartheta'_4(2id)}{\vartheta_4(2id)}\right]$.

(γ) If we take two barriers at a fixed distance apart, *i.e.*, if c is constant, then when b is vanishingly small the system is stable when and only when $a = 0{\cdot}281b$. This corresponds with the VON KÁRMÁN result. As b increases, the stable cases are obtained by increasing a almost proportionately. This continues till we get to the case $b = 0{\cdot}815c$, $a = 0{\cdot}256b = 0{\cdot}208c$. The curve showing the relationship between μ ($\equiv \pi b/2c$) and ν ($\equiv \pi a/2c$) for these cases is called the "Stability Curve." For every value of b greater than $0{\cdot}815c$, we get a range of values of a in which the system is stable. Ultimately, when $b \geq 1{\cdot}419c$ (*i.e.*, $t \leq 1{\cdot}409$) the system is stable for all values of a. The total domain of stability is referred to as the "Stability Area."

(δ) When c is infinite, while a and b remain finite, the relative stream lines are similar to those of fig. 4 (g). When $0 < b/c < 1{\cdot}155$, the change in the form of the stream lines as a increases from 0 to c, is given by the sequence of diagrams in fig. 4. When $1{\cdot}155 = b/c$, $1{\cdot}155 < b/c < 1{\cdot}419$, $1{\cdot}419 \leq b/c$, the corresponding changes are given by figs. 5, 6, 7, respectively. As for the form of the relative stream lines in the stable cases, we get that when the barriers are at an infinite distance apart, the stream lines are similar to those of fig. 4 (g)—thus agreeing with VON KÁRMÁN's result (*loc. cit.*, p. 53). When $b/c < 1 \cdot 419$ there appears to be no well defined connection between stability and the form of the stream lines. The type of stream line changes in a continuous but rather complicated manner (see fig. 9). When $b/c \geq 1 \cdot 419$, the stream lines are always of the form given in fig. 7 and, as stated above, the system is always stable.

Problem II.

(α) The ω function is $-\dfrac{i\kappa}{2\pi} \log \dfrac{\vartheta_1 (\mathrm{Z} - id)}{\vartheta_1 (\mathrm{Z} + id)}$ where $\tau = i\,\dfrac{c}{b}$.

(β) All the vortices move forward with a velocity $\dfrac{i\kappa}{4\pi b} \dfrac{\vartheta'_1 (2id)}{\vartheta_1 (2id)}$.

(γ) The system is always unstable.

(δ) The change in the form of the stream lines as a changes from 0 to c is always given by the sequence of diagrams in fig. 11.

PROBLEM I.

2. *Problem Defined.*

2.1. The system of vortices is that of a "KÁRMÁN street" between two plane barriers. There are positive vortices at the points $(2nb,\ a)$ and negative vortices at the points $((2n - 1)\ b,\ -a)$ where n assumes all integral values from $-\infty$ to $+\infty$. To obtain a system in which the barriers, $y = \pm\, c$, are stream lines, it is necessary to introduce an infinite series of image vortices for each vortex of the "street." The velocity of a vortex due to its rotations round two vortices which are equidistant from it, and which

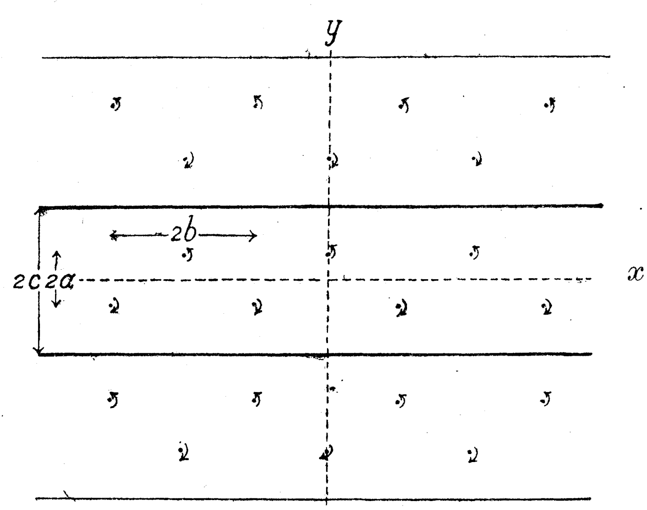


FIG. 2.

lie on a line parallel to the x axis, is a velocity parallel to the x axis. Hence the system of vortices will move with a uniform velocity parallel to the x axis, for the velocity of any vortex of the "street" is the resultant of a doubly infinite series of such components, plus the velocity due to its own line of image vortices, which is also parallel to the x axis.

2.2. *Determination of the* ω *Function.*

2.21. The following notation will be adopted

$$\omega = \phi + i\psi.$$

The components of velocity at the point (x, y) are defined to be $u(x, y)$ and $v(x, y)$ where

$$u(x, y) = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v(x, y) = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

When ψ is considered as a function of position it will be referred to as ψ (X, Y).

2.22. The ω function for a positive vortex of strength κ at the point α is $-\frac{i\kappa}{2\pi}\log(z-\alpha)$. The ω function for the system under consideration will have to be such that it is of the form

$$-\frac{i\kappa}{2\pi}\log(z-\alpha) + \text{regular function of } z$$

in the neighbourhood of a positive vortex, and of the form

$$+\frac{i\kappa}{2\pi}\log(z-\alpha) + \text{regular function of } z$$

in the neighbourhood of a negative vortex. The ω function will therefore have to be $-\frac{i\kappa}{2\pi}\log f(z)$ where $f(z)$ has simple zeros at the positive vortices and simple poles at the negative vortices. The "zeros" can be divided into two groups

$$\text{Z} = z/2b = ia/2b + n + m\, i4c/2b = id + n + m\tau,$$
$$\text{Z} = z/2b = ia/2b + (n + \tfrac{1}{2}) + (m + \tfrac{1}{2})\, i4c/2b = id + n + m\tau + \tfrac{1}{2} + \tfrac{1}{2}\tau,$$

where m and n assume all integral values from $-\infty$ to $+\infty$. Similarly the "poles" can be divided into two groups

$$\text{Z} = z/2b = -i\,a/2b + (n + \tfrac{1}{2}) + m\, i4c/2b = -id + n + m\tau + \tfrac{1}{2},$$
$$\text{Z} = z/2b = -i\,a/2b + n + (m + \tfrac{1}{2})\, i4c/2b = -id + n + m\tau + \tfrac{1}{2}\tau.$$

This symmetrical disposition of the "zeros" and "poles" suggests the introduction of

ϑ functions for it is well known that $\vartheta_1(v)$, $\vartheta_2(v)$, $\vartheta_3(v)$, $\vartheta_4(v)$ have simple zeros at the points $(n + m\tau)$, $(n + m\tau + \frac{1}{2})$, $(n + m\tau + \frac{1}{2} + \frac{1}{2}\tau)$, $(n + m\tau + \frac{1}{2}\tau)$ respectively. We see immediately that the $f(\mathrm{Z})$ for our problem is

$$\frac{\vartheta_1(\mathrm{Z} - id)\,\vartheta_3(\mathrm{Z} - id)}{\vartheta_2(\mathrm{Z} + id)\,\vartheta_4(\mathrm{Z} + id)},$$

2.23. This expression could have been obtained from other considerations as follows: Neglecting additive constants, the ω function for a positive vortex at the point $(2rb, a)$ between the barriers $y = \pm c$ is

$$-\frac{i\kappa}{2\pi}\log\frac{\prod_{n=-\infty}^{n=+\infty}(z - 2rb - i(4nc + a))}{\prod_{m=-\infty}^{m=+\infty}(z - 2rb - i(\overline{4m - 2}\,c - a))}$$

$$= -\frac{i\kappa}{2\pi}\log(z - 2rb - ia)\frac{\prod_{n=1}^{n=\infty}[(z - 2rb - ia)^2 + 16n^2c^2]}{\prod_{m=1}^{m=\infty}[(z - 2rb + ia)^2 + 4(2m - 1)^2c^2]}$$

$$= -\frac{i\kappa}{2\pi}\log\frac{\sinh[(z - 2rb - ia)\pi/4c]}{\cosh[(z - 2rb + ia)\pi/4c]}.$$

Therefore the ω function for the bounded "KÁRMÁN street" system is

$$-\frac{i\kappa}{2\pi}\log\left\{\frac{\prod_{r=-\infty}^{r=+\infty}\sinh[(z - 2rb - ia)\pi/4c]\prod_{s=-\infty}^{s=+\infty}\cosh[(z - (2s - 1)b - ia)\pi/4c]}{\prod_{r=-\infty}^{r=+\infty}\cosh[(z - 2rb + ia)\pi/4c]\prod_{s=-\infty}^{s=+\infty}\sinh[(z - (2s - 1)b + ia)\pi/4c]}\right\}$$

$$= -\frac{i\kappa}{2\pi}\log\left\{\frac{\sinh[(z - ia)\pi/4c]\prod_1^\infty(\cosh 2rb\pi/2c - \cosh(z - ia)\pi/2c)\prod_1^\infty(\cosh\overline{2s - 1}\,b\pi/2c + \cosh(z - ia)\pi/2c)}{\cosh[(z + ia)\pi/4c]\prod_1^\infty(\cosh 2rb\pi/2c + \cosh(z + ia)\pi/2c)\prod_1^\infty(\cosh\overline{2s - 1}\,b\pi/2c - \cosh(z + ia)\pi/2c)}\right.$$

In this expression put $q_1 = \exp(-\pi b/2c)$, $z/2b = \mathrm{Z}$, $a/2b = d$; we get

$$= -\frac{i\kappa}{2\pi}\log\frac{\sin[\pi(\mathrm{Z} - id)/\tau]\prod_1^\infty(1 - 2q_1^{2r}\cos 2\pi(\mathrm{Z} - id)/\tau + q_1^{4r})\prod_1^\infty(1 + 2q_1^{2s-1}\cos 2\pi(\mathrm{Z} - id)/\tau + q_1^{4s-2})}{\cos[\pi(\mathrm{Z} + id)/\tau]\prod_1^\infty(1 + 2q_1^{2r}\cos 2\pi(\mathrm{Z} + id)/\tau + q_1^{4r})\prod_1^\infty(1 - 2q_1^{2s-1}\cos 2\pi(\mathrm{Z} + id)/\tau + q_1^{4s-2})}$$

$$= -\frac{i\kappa}{2\pi}\log\frac{\vartheta_1[(\mathrm{Z} - id)/\tau\,|-1/\tau]\,\vartheta_3[(\mathrm{Z} - id)/\tau\,|-1/\tau]}{\vartheta_2[(\mathrm{Z} + id)/\tau\,|-1/\tau]\,\vartheta_4[(\mathrm{Z} + id)/\tau\,|-1/\tau]}$$

$$= -\frac{i\kappa}{2\pi}\log\exp\left(\frac{8\pi d\mathrm{Z}}{\tau}\right)\frac{\vartheta_1[\mathrm{Z} - id\,|\,\tau]\,\vartheta_3[\mathrm{Z} - id\,|\,\tau]}{\vartheta_2[\mathrm{Z} + id\,|\,\tau]\,\vartheta_4[\mathrm{Z} + id\,|\,\tau]}$$

$$= -\frac{\kappa d}{c}z - \frac{i\kappa}{2\pi}\log f(\mathrm{Z}).$$

2.24. The term $-\frac{\kappa d}{c} z$ means that a velocity $-\frac{\kappa d}{c}$ has been impressed on the whole system. This does not alter the shape of the *relative* stream lines, and so we may use $f(\mathrm{Z})$ for this problem.

3. *Uniqueness of* $f(\mathrm{Z})$.

3.1. We have

$$\omega = \phi + i\psi = -\frac{i\kappa}{2\pi} \log f(\mathrm{Z}),$$

therefore

$$f(\mathrm{Z}) = \exp\left(-\frac{2\pi\psi}{\kappa} + i\frac{2\pi\phi}{\kappa}\right),$$

that is

$$|f(\mathrm{Z})| = \exp\left(-\frac{2\pi\psi}{\kappa}\right).$$

The stream lines of the system are therefore given by

$$\left|\frac{\vartheta_1(\mathrm{Z}-id)\,\vartheta_3(\mathrm{Z}-id)}{\vartheta_2(\mathrm{Z}+id)\,\vartheta_4(\mathrm{Z}+id)}\right| = \text{constant}.$$

Let us denote this constant by $[\mathrm{C}(\mathrm{X}, \mathrm{Y})]^{\frac{1}{2}}$ since it is a function of position. One of the conditions of this problem is that the barriers are stream lines of the system and therefore $|f(\mathrm{Z})|$ must be constant along each of the lines $y = \pm c$. It is easy to verify that in the case of the single vortex between the barriers, $|f(\mathrm{Z})| = 1$ on both barriers and in the case of the bounded " KÁRMÁN street."

$$\mathrm{C}(\mathrm{X}, \tfrac{1}{4}t) = \exp(-4\pi d) \quad \text{and} \quad \mathrm{C}(\mathrm{X}, -\tfrac{1}{4}t) = \exp(4\pi d).$$

The fact that C, and therefore ψ, has different values on the two barriers indicates that across any line drawn perpendicular to the barriers and terminated by them, there is a mean flow of $\psi(\mathrm{X}, \frac{1}{4}t) - \psi(\mathrm{X}, -\frac{1}{4}t)$. If, however, we use the function $\mathrm{F}(\mathrm{Z})$ where $\mathrm{F}(\mathrm{Z}) = \exp\left(\frac{8\pi d\mathrm{Z}}{\tau}\right).f(\mathrm{Z})$, it is easy to see that $|\mathrm{F}(\mathrm{X}+\frac{1}{4}\tau)| = |\mathrm{F}(\mathrm{X}-\frac{1}{4}\tau)| = 1.$

3.2. In spite of the fact that $f(\mathrm{Z})$ has been obtained rather arbitrarily, it can be shown that it is the unique function for the problem under consideration—or rather, any other function that satisfies the conditions of the problem is of the form $\exp(l\mathrm{Z}+m) f(\mathrm{Z})$ where l and m are constants. As explained previously, the introduction of the exponential does not alter the shape of the *relative* stream lines.

3.21. The conditions that $f(\mathrm{Z})$ must satisfy, when expressed mathematically, become :—

(1) $f(\mathrm{Z})$ is regular in the strip $-c \leq y \leq c$ and has simple poles and zeros at the negative and positive vortices respectively.

(2) $|f(\mathrm{Z})|$ is constant over each barrier.

(3) The shape of a stream line is a continuous repetition of that portion of the stream line between the lines $x = 0$ and $x = 2b$.

This third condition really means

$$u(x + 2b, y) = u(x, y) \quad \text{and} \quad v(x + 2b, y) = v(x, y)$$

that is

$$\psi(X + 1, Y) = \psi(X, Y) + \text{constant.}$$

However, since there can be no mean flow across the straight line joining $(z + 2b)$ and z, we must have $\psi(X + 1, Y) = \psi(X, Y)$ so that the above constant is zero. Also, since $\exp(-2\pi\psi/\kappa) = |f(Z)|$, it appears that condition (3) above means that $|f(Z)|$ has the period 1. It is now necessary to show the sufficiency of this criterion—that is, if $|f(Z)|$ has the period 1 then $u(x + 2b, y) = u(x, y)$, etc. We have

$$1 = \frac{|f(Z+1)|}{|f(Z)|} = \exp\left(-\frac{2\pi}{\kappa}\psi(X + 1, Y)\right)\Big/\exp\left(-\frac{2\pi}{\kappa}\psi(X, Y)\right)$$

$$= \exp\left(-\frac{2\pi}{\kappa}[\psi(X + 1, Y) - \psi(X, Y)]\right),$$

that is

$$\psi(X + 1, Y) = \psi(X, Y).$$

The rest follows immediately.

3.22. Let $g(Z)$ be some function other than $f(Z)$ which satisfies the conditions of the problem. Further let $|g(X + \frac{1}{4}\tau)| = C_1$ and $|g(X - \frac{1}{4}\tau)| = C_2$. Let us introduce the functions $F(Z)$ and $G(Z)$ where

$$F(Z) = \exp\left(\frac{8\pi dZ}{\tau}\right)f(Z) \quad \text{and} \quad G(Z) = \exp\left(\frac{2Z}{\tau}\log\left(\frac{C_2}{C_1}\right) - \tfrac{1}{2}\log C_1C_2\right)g(Z).$$

It can easily be verified that $|F(Z)|$ and $|G(Z)|$ are unity on each of the barriers and also have the period 1. Now let $\Phi(Z) = G(Z)/F(Z)$. $\Phi(Z)$ is regular in $-c \lessgtr y \leq c$ and has no zeros in this strip. $|\Phi(Z)|$ is 1 on both barriers and has the period 1. It is necessary to prove that $\Phi(Z)$ is a constant. We have, $\Phi(Z)$ regular, and therefore bounded, in the domain $-b \leq x \leq b, -c \leq y \leq c$. By periodicity it is bounded in the whole strip $-c \leq y \leq c$. Since $|\Phi(Z)|$ is 1 on the boundaries, we get, by a PHRAGMÉN-LINDELÖF* theorem that $|\Phi(Z)| \leq 1$ inside this strip. But $[1/\Phi(Z)]$ also has the same properties so that $|1/\Phi(Z)| \leq 1$ inside. Thus $|\Phi(Z)| = 1$ throughout the strip and $\log \Phi(Z)$ is therefore regular. If $\log \Phi(Z) = r + is$ then its real part r is zero and therefore $\frac{\partial r}{\partial x} = \frac{\partial r}{\partial y} = 0$. Hence $\frac{\partial s}{\partial x} = \frac{\partial s}{\partial y} = 0$ by the CAUCHY-RIEMANN Differential

* 'Acta Mathematica,' vol. 31, pp. 381–406 (1908).

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Equations. Therefore s is a constant, giving log $\Phi(Z)$ and therefore $\Phi(Z)$ as constants, *i.e.*,

$$G(Z)/F(Z) = \text{constant},$$

i.e.,

$$g(Z) = \exp(lZ + m) f(Z),$$

where l and m are some constants.

4. *Explicit Equation for the Stream Lines.*

4.1. In our present problem, where τ is purely imaginary, we have

$$|\vartheta_\alpha^2(x+iy)| = |\vartheta_\alpha(x+iy)\,\vartheta_\alpha(x-iy)|, \qquad \alpha = 1, 2, 3, 4.$$

Inserting in the formulæ given in 'Tannery and Molk,' vol. 2, p. 159, we get:

$$\vartheta_3^2(0)\,|\vartheta_1^2(x+iy)| = \vartheta_3^2(iy)\,\vartheta_1^2(x) - \vartheta_1^2(iy)\,\vartheta_3^2(x),$$

$$\vartheta_3^2(0)\,|\vartheta_2^2(x+iy)| = -\vartheta_4^2(iy)\,\vartheta_1^2(x) + \vartheta_2^2(iy)\,\vartheta_3^2(x),$$

$$\vartheta_3^2(0)\,|\vartheta_3^2(x+iy)| = \vartheta_1^2(iy)\,\vartheta_1^2(x) + \vartheta_3^2(iy)\,\vartheta_3^2(x),$$

$$\vartheta_3^2(0)\,|\vartheta_4^2(x+iy)| = \vartheta_2^2(iy)\,\vartheta_1^2(x) + \vartheta_4^2(iy)\,\vartheta_3^2(x).$$

Therefore

$$\exp\left(-\frac{4\pi\psi}{\kappa}\right)$$

$$\equiv C(X, Y) = \frac{f_1 f_2}{f_3 f_4}$$

$$\equiv \frac{[\vartheta_3^2(i\overline{Y-d})\,\vartheta_1^2(X) - \vartheta_1^2(i\overline{Y-d})\,\vartheta_3^2(X)][\vartheta_1^2(i\overline{Y-d})\,\vartheta_1^2(X) + \vartheta_3^2(i\overline{Y-d})\,\vartheta_3^2(X)]}{[-\vartheta_4^2(i\overline{Y+d})\,\vartheta_1^2(X) + \vartheta_2^2(i\overline{Y+d})\,\vartheta_3^2(X)][\vartheta_2^2(i\overline{Y+d})\,\vartheta_1^2(X) + \vartheta_4^2(i\overline{Y+d})\,\vartheta_3^2(X)]}.$$

The terms involving X and Y can be separated, giving:

$$\frac{\vartheta_1^2(X)}{\vartheta_3^2(X)} - \frac{\vartheta_3^2(X)}{\vartheta_1^2(X)} = \frac{[\vartheta_1^4(i\overline{Y-d}) - \vartheta_3^4(i\overline{Y-d})] + C(X, Y)[\vartheta_2^4(i\overline{Y+d}) - \vartheta_4^4(i\overline{Y+d})]}{[\vartheta_1^2(i\overline{Y-d})\,\vartheta_3^2(i\overline{Y-d})] + C(X, Y)[\vartheta_2^2(i\overline{Y+d})\,\vartheta_4^2(i\overline{Y+d})]}.$$

4.2. If now we impress a velocity $-U$ on the whole system, we must put

$$\omega = -Uz - \frac{i\kappa}{2\pi}\log f(Z) = -\frac{i\kappa}{2\pi}\log\left\{\exp\left(-i\frac{2\pi U}{\kappa}z\right) f(Z)\right\}.$$

The equations of the stream lines are

$$\exp\left(-\frac{4\pi\psi}{\kappa}\right) = \text{constant} = S(X, Y) = \exp\left(\frac{8\pi b U Y}{\kappa}\right)\frac{f_1 f_2}{f_3 f_1}.$$

This equation becomes

$$\frac{\vartheta_1^2(\mathrm{X})}{\vartheta_3^2(\mathrm{X})}-\frac{\vartheta_3^2(\mathrm{X})}{\vartheta_1^2(\mathrm{X})}$$

$$=\frac{[\vartheta_1^4(i\overline{\mathrm{Y}-d})-\vartheta_3^4(i\overline{\mathrm{Y}-d})]+\mathrm{S}(\mathrm{X},\mathrm{Y})\exp\left(-\frac{8\pi b\mathrm{UY}}{\kappa}\right)[\vartheta_2^4(i\overline{\mathrm{Y}+d})-\vartheta_4^4(i\overline{\mathrm{Y}+d})]}{[\vartheta_1^2(i\overline{\mathrm{Y}-d})\,\vartheta_3^2(i\overline{\mathrm{Y}-d})]+\mathrm{S}(\mathrm{X},\mathrm{Y})\exp\left(-\frac{8\pi b\mathrm{UY}}{\kappa}\right)[\vartheta_2^2(i\overline{\mathrm{Y}+d})\,\vartheta_4^2(i\overline{\mathrm{Y}+d})]}\quad . \quad (1)$$

In order to obtain the relative stream lines we put U equal to the velocity of the vortices. This is

$$\operatorname*{Lt}_{z\to ia}\left|\frac{d}{dz}\left[-\frac{i\kappa}{2\pi}\log f(\mathrm{Z})+\frac{i\kappa}{2\pi}\log(\mathrm{Z}-id)\right]\right|=\frac{i\kappa}{4\pi b}\left[\frac{\vartheta'_2(2id)}{\vartheta_2(2id)}+\frac{\vartheta'_4(2id)}{\vartheta_4(2id)}\right]. \quad . \quad (2)$$

5. *Discussion of* U_d (*Velocity of Vortices*).

Let U_d represent the expression (2) and let us consider it more fully. We see that*

$$\mathrm{U}_d=\frac{\kappa}{4b}\left[\tanh 2\pi d-\sum_{r=1}^{r=\infty}\frac{4q^r}{1-(-q)^r}\sinh 4r\pi d\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When $c\to\infty$, that is $q\to 0$, we get

$$\mathrm{U}_d=\frac{\kappa}{4b}\tanh 2\pi d=\frac{\kappa}{4b}\tanh\pi\frac{a}{b},$$

a result given by LAMB and VON KÁRMÁN (*loc. cit.*). This corresponds to the unbounded "KÁRMÁN street." We have in addition

$$\frac{\partial^2}{\partial d^2}\mathrm{U}_d=-\frac{2\kappa\pi^2}{b}\left[\operatorname{sech}^2 2\pi d\tanh 2\pi d+\sum_{r=1}^{r=\infty}\frac{8r^2q^r}{1-(-q)^r}\sinh 4r\pi d\right].$$

Since $q<1$, this is always negative, except at $d=0$, where it is zero. Hence the curve U_d has a point of inflexion at $d=0$ and has its convexity upwards. But $\mathrm{U}_d=0$ and $\mathrm{U}_{\frac{1}{4}t}=-\infty$. Hence U_d always has a zero at $d=0$, and it has an additional one if and only if $\left[\frac{\partial}{\partial d}\mathrm{U}_d\right]_{d=0}>0$. Now

$$\left[\frac{\partial}{\partial d}\mathrm{U}_d\right]_{d=0}=-\frac{\kappa}{2\pi b}\left[\frac{\vartheta''_2(0)}{\vartheta_2(0)}+\frac{\vartheta''_4(0)}{\vartheta_4(0)}\right].$$

The terms involved in this expression have been tabulated by NAGAOKA and SAKURAI.† They have been tabulated against $k^2(\equiv\vartheta_2^4(0)/\vartheta_3^4(0))$ in the range $0\leq k^2\leq\frac{1}{2}$. The following formulæ enable determinations to be made in the whole range $0\leq k^2\leq 1$.

* TANNERY and MOLK, vol. 4, p. 100.

† 'Inst. Phys. Chem. Res. Tokyo,' vol. 2, p. 1 (1922).

Putting $\tau_1 = -1/\tau$, $\log_e q = i\pi\tau$, $\log_e q_1 = i\pi\tau_1$, we get from the transformation formulæ

$$\left.\begin{aligned} \tau\left\{\frac{\vartheta''_2(0\,|\,\tau)}{\vartheta_2(0\,|\,\tau)}+\frac{\vartheta''_4(0\,|\,\tau)}{\vartheta_4(0\,|\,\tau)}\right\}+\tau_1\left\{\frac{\vartheta''_2(0\,|\,\tau_1)}{\vartheta_2(0\,|\,\tau_1)}+\frac{\vartheta''_4(0\,|\,\tau_1)}{\vartheta_4(0\,|\,\tau_1)}\right\}+4i\pi=0 \\ \left\{\frac{\vartheta''_2(0\,|\,\tau)}{\vartheta_2(0\,|\,\tau)}+\frac{\vartheta''_4(0\,|\,\tau)}{\vartheta_4(0\,|\,\tau)}\right\}\log_e q+\left\{\frac{\vartheta''_2(0\,|\,\tau_1)}{\vartheta_2(0\,|\,\tau_1)}+\frac{\vartheta''_4(0\,|\,\tau_1)}{\vartheta_4(0\,|\,\tau_1)}\right\}\log_e q_1=4\pi^2 \end{aligned}\right\}. \quad (4)$$

We find that $\left[\dfrac{\partial}{\partial d}\mathrm{U}_d\right]_{d=0} \gtreqless 0$ according as $k^2 \gtreqless 0\cdot826$, *i.e.*, $t \gtreqless 0\cdot7096$. Hence U_d has a zero, other than the one at $d = 0$, if $t > 0\cdot7096$; and only has the zero at $d = 0$ if $t \leq 0\cdot7096$.

6. Returning to the stream lines we see that although the shape of the stream lines is a continuous repetition of the configuration in the area $-b \leq x \leq b$, $-c \leq y \leq c$, it is not necessary to discuss the value of S (X, Y) over the whole of this area. It is sufficient to know the value of S (X, Y) in the region $0 \leq x \leq b$, $0 \leq y \leq c$ for the following reasons :—

(1) S (X, Y) is symmetrical with respect to the Y axis.

(2) $\mathrm{S}(\mathrm{X}, \mathrm{Y}) \times \mathrm{S}(\frac{1}{2} - \mathrm{X}, -\mathrm{Y}) = 1$. This means that if S (X, Y) is a stream line, then $\mathrm{S}(\frac{1}{2} - \mathrm{X}, -\mathrm{Y})$ is also a stream line.

A particular case of the second consideration is seen in

$$\mathrm{S}(\mathrm{X}, \tfrac{1}{4}t) = \exp\frac{4\pi c}{\kappa}\left(\mathrm{U}_d - \frac{\kappa d}{c}\right); \quad \mathrm{S}(\mathrm{X}, -\tfrac{1}{4}t) = \exp-\frac{4\pi c}{\kappa}\left(\mathrm{U}_d - \frac{\kappa d}{c}\right).$$

Also if we put $\mathrm{X} = \frac{1}{4}$ and $\mathrm{Y} = 0$, we get $\mathrm{S}(\frac{1}{4}, 0) = 1$.

7. *Points of Zero Velocity.*

7.1. Reverting to equation (1) and putting $\vartheta_1^2(\mathrm{X})/\vartheta_3^2(\mathrm{X}) = l$, we get

$$l - 1/l = (\text{function of } y \text{ and } \mathrm{S}(\mathrm{X}, \mathrm{Y})).$$

Along a line parallel to the axis of x we would have

$$l - 1/l = (\text{function of } \mathrm{S}(\mathrm{X}, \mathrm{Y}) \text{ only}) = k \text{ (say)},$$

that is

$$l^2 - lk - 1 = 0,$$

and k is a one valued function of S (X, Y).

If l_1 and l_2 are the roots of this equation then $l_1 l_2 = -1$. But of necessity both l_1 and l_2 must be positive. Hence only one root of this quadratic can be used. We know also that a one to one correspondence exists between X and l if $0 \leq \mathrm{X} \leq \frac{1}{2}$, and this means that on a line parallel to the axis of x there cannot be two points giving the same S (X, Y); that is, the stream line S (X, Y) = constant, cannot cut the line y = constant, in more than one point.

7.2. An immediate corollary of this is that there can be no point of zero velocity in the area bounded by the barriers and the lines $X = \frac{1}{2}$, $X = 0$. The reason for this is that there can be no point of zero velocity in the system except at those points where a stream line has a node—and the line parallel to the x axis through this node would cut the stream line in two points. Points of zero velocity may occur on the boundaries or on $X = \frac{1}{2}$ and $X = 0$.

7.21. This and other results can be deduced from more general considerations as follows:

We have

$$\omega = -\mathrm{U}z - \frac{i\kappa}{2\pi}\log f(\mathrm{Z}),$$

that is

$$\left|\frac{d\omega}{dz}\right| = \left|-\mathrm{U} - \frac{i\kappa}{4\pi b}\frac{f'(\mathrm{Z})}{f(\mathrm{Z})}\right| = \text{velocity}.$$

But $f(\mathrm{Z}+1) = f(\mathrm{Z})$ and $f(\mathrm{Z}+\tau) = \exp(-8\pi d) f(\mathrm{Z})$, so that

$$\frac{f'(\mathrm{Z}+\tau)}{f(\mathrm{Z}+\tau)} = \frac{f'(\mathrm{Z}+1)}{f(\mathrm{Z}+1)} = \frac{f'(\mathrm{Z})}{f(\mathrm{Z})}.$$

$d\omega/dz$ is doubly periodic and is therefore an elliptic function. Its periods are 1 and τ. $f(\mathrm{Z})$ has two simple poles and two simple zeros in a period parallelogram so that $f'(\mathrm{Z})/f(\mathrm{Z})$, and therefore $d\omega/dz$, has four simple poles in a period parallelogram. Since $d\omega/dz$ is an elliptic function it must also have four zeros in this domain.

7.22. If a zero were to occur at the point (x, y) then by symmetry there would have to be zeros at the points (x, y), $(x, 2c-y)$, $(2b-x, y)$, $(2b-x, 2c-y)$, $(b \pm x, -y)$, $(b \pm x, 2c+y)$, that is, there would have to be eight zeros within a period parallelogram —which is impossible. The statement about the symmetrical disposition of the points of zero velocity is obvious from the point of view of hydrodynamics and it can be seen from the point of view of analysis, firstly, from the fact that $\dfrac{f'(\mathrm{Z})}{f(\mathrm{Z})} = \dfrac{f'(\frac{1}{2}-\mathrm{Z})}{f(\frac{1}{2}-\mathrm{Z})}$, and, secondly, from the fact that since $d\omega/dz$ is real on lines such as $X = 0$, then by "SCHWARZ's Reflection Principle" it takes conjugate values at points which are image points with respect to this line.

7.23. There can be four zeros in one of the following ways:—

(1) $x = 0$, in which case the zeros are at $(0, y)$, $(0, 2c-y)$, $(b, -y)$, $(b, 2c+y)$.

(2) $y = -c$, in which case the zeros are at $(x, -c)$, $(b \pm x, c)$, $(2b-x, -c)$.

(3) $x = \frac{1}{2}b$, $y = 0$, in which case the zeros are at $(\frac{1}{2}b, 0)$, $(\frac{1}{2}b, 2c)$, $(\frac{3}{2}b, 0)$, $(\frac{3}{2}b, 2c)$.

This third case is impossible for the following reasons :—

We have

$$f(Z)\, f(\tfrac{1}{2} - Z) = 1,$$

so that

$$\frac{f'(Z)}{f(Z)} - \frac{f'(\frac{1}{2} - Z)}{f(\frac{1}{2} - Z)} = 0, \quad i.e., \quad f_1(Z) - f_1(\tfrac{1}{2} - Z) = 0,$$

and

$$f_1'(Z) + f_1'(\tfrac{1}{2} - Z) = 0, \qquad \text{where} \qquad f_1'(Z) = \frac{d}{dZ}\left(\frac{f'(Z)}{f(Z)}\right).$$

Putting $Z = \frac{1}{4}$ we see

$$f_1'(\tfrac{1}{4}) = 0, \qquad i.e., \qquad \left[\frac{d}{dZ}\left(\frac{f'(Z)}{f(Z)}\right)\right]_{Z=\frac{1}{4}} = 0.$$

But

$$\left(\frac{d^2\omega}{dz^2}\right)_{z=\frac{1}{2}b} = -\frac{i\kappa}{8\pi b^2}\left[\frac{d}{dZ}\left(\frac{f'(Z)}{f(Z)}\right)\right]_{z=\frac{1}{4}} = 0.$$

This means that if $d\omega/dz$ is zero at $Z = \frac{1}{4}$, then this zero is of the second order, thus making eight zeros within a period parallelogram—which is impossible. Cases (1) and (2) may be combined in the statement that in any configuration of vortices there is one and only one zero in the domain $-c \leq y < c$, $0 \leq x < b$, and this zero must occur on one of the lines $x = 0$ or $y = -c$. It is shown in paragraph 8.4 that $u(0, y)$ is always negative when $Y > d$, and always positive when $d > Y \geq 0$; so we have that in any system of vortices there must be one and only one point of zero velocity within the domain $-c \leq y < c$; $0 \leq x < b$, and that this point must be on one of the lines $x = 0$, $0 > y \geq -c$, and $y = -c$.

8. *The Stream Lines Relative to the Vortices.*

8.1. Several interesting results can be foreseen by considering the functional form of $S(X, Y)$. For instance, when $d = 0$ it can easily be verified that the line $X = \frac{1}{4}$ is a stream line for all values of t. This means that when $d = 0$ the stream lines are always of the form indicated in fig. 4 (*a*).

8.2. When a is very nearly equal to c, the stream lines in the neighbourhood of the vortex at $(0, a)$ should be similar to those in the vicinity of a "vortex pair,"* since here the effect of the other vortices may be neglected in comparison with the effect of the vortex at $(0, a)$ and its image vortex at $(0, 2c - a)$. Hence when a is very nearly equal to c the stream lines must always be similar to those of fig. 4 (*h*).

8.3. When $a = c$, we have $C(X, Y) = \exp(-4\pi Y)$ so that the general stream lines are the straight lines $y =$ constant. The relative stream lines are given by

$$S(X, Y) = \exp\left(\frac{8\pi b U Y}{\kappa} - 4\pi Y\right),$$

* LAMB, 'Hydrodynamics,' p. 204.

which is indeterminate since $U = -\infty$. For every value of t therefore, the form of the stream lines must change in some continuous manner from that of fig. 4 (a) to that of fig. 4 (h), and finally to that in which there is no motion at all.

8.4. *Discussion of* $u\,(0, y)$ *in the range* $c \geq y \geq 0$.

We have

$$S\,(X, Y) = \exp\left(-\frac{4\pi\psi}{\kappa}\right).$$

Hence

$$u\,(x, y) = -\frac{\kappa}{8\pi b}\frac{1}{S\,(X, Y)}\frac{\partial}{\partial Y}S\,(X, Y) \quad \text{and} \quad v\,(x, y) = \frac{\kappa}{8\pi b}\frac{1}{S\,(X, Y)}\frac{\partial}{\partial X}S\,(X, Y).$$

We have immediately $v\,(0, y) = 0$, and also

$$u\,(0, y) = -\frac{\kappa i}{4\pi b}\left[\frac{\vartheta'_2\,(2id)}{\vartheta_2\,(2id)} + \frac{\vartheta'_4\,(2id)}{\vartheta_4\,(2id)} + \frac{\vartheta'_1\,(i\,\overline{Y-d})}{\vartheta_1\,(i\,\overline{Y-d})} + \frac{\vartheta'_3\,(i\,\overline{Y-d})}{\vartheta_3\,(i\,\overline{Y-d})} - \frac{\vartheta'_2\,(i\,\overline{Y+d})}{\vartheta_2\,(i\,\overline{Y+d})} - \frac{\vartheta'_4\,(i\,\overline{Y+d})}{\vartheta_4\,(i\,\overline{Y+d})}\right]$$

$$= -\frac{\kappa}{4b}\left[\tanh 2\pi d - \sum_{p=1}^{p=\infty}\frac{4q^p}{1-(-q)^p}\sinh 4p\pi d + \cosh \pi\,(Y-d) - \sum_{p=1}^{p=\infty}\frac{4\,(-q)^p}{1-(-q)^p}\sinh 2p\pi\,(Y-d) - \tanh \pi\,(Y+d) + \sum_{p=1}^{p=\infty}\frac{4q^p}{1-(-q)^p}\sinh 2p\pi\,(Y+d)\right]$$

$$= -\frac{\kappa}{4b}\left[\frac{2+\sinh 2\pi d\,(\sinh 2\pi Y + \sinh 2\pi d)}{\cosh 2\pi d\,(\sinh 2\pi Y - \sinh 2\pi d)} + \Sigma\,\frac{8q^r\cosh 2r\pi d}{1+q^r}(\sinh 2r\pi Y - \sinh 2r\pi d) + \Sigma\,\frac{8q^s\sinh 2s\pi d}{1-q^s}(\cosh 2s\pi Y - \cosh 2s\pi d)\right],$$

where r assumes the values of all the odd integers, and s the even integers, from 1 to ∞. If $Y > d$, $u\,(0, y)$ is negative and the streaming above $Y = d$ is always to the left. If $d > Y \geq 0$, $u\,(0, y)$ is positive, so that the streaming in the range $d > Y \geq 0$ is always to the right.

8.5. *Discussion of* $u\,(0, -c)$.

We have

$$u\,(0, -c) = -\frac{\kappa i}{4\pi b}\left[\frac{\vartheta'_2\,(2id)}{\vartheta_2\,(2id)} + \frac{\vartheta'_4\,(2id)}{\vartheta_4\,(2id)} + \frac{2\vartheta'_2\,(\frac{1}{4}\tau - id)}{\vartheta_2\,(\frac{1}{4}\tau - id)} + \frac{2\vartheta'_4\,(\frac{1}{4}\tau - id)}{\vartheta_4\,(\frac{1}{4}\tau - id)} + 2i\pi\right],$$

that is

$$u(0, -c) = -\phi(d) - 2\phi(\tfrac{1}{8}t - \tfrac{1}{2}d),$$

where

$$\phi(d) = U_d - \kappa d/c.$$

Since U_d is a function that is convex upwards, we see that $u(0, -c)$ is concave upwards and has therefore no zeros or two zeros. The critical case occurs when $u(0, -c)$ has two coincident zeros. In this case

$$\phi(d) + 2\phi(\tfrac{1}{8}t - \tfrac{1}{2}d) = 0, \quad \ldots \ldots \quad (5)$$

and

$$\phi'(d) - \phi'(\tfrac{1}{8}t - \tfrac{1}{2}d) = 0. \quad \ldots \ldots \quad (6)$$

Since $\phi(d)$ is always convex upwards, $\phi'(d)$ is a one valued function of d, so that if equation (6) is to be satisfied then

$$d = \tfrac{1}{8}t - \tfrac{1}{2}d, \quad i.e., \quad d = \tfrac{1}{12}t.$$

Substituting in (5) we get $\phi(\frac{1}{12}t) = 0$, so that the limiting case is given by the solution of

$$\frac{i\kappa}{4\pi b}\left[\frac{\vartheta'_2(\frac{1}{6}\tau)}{\vartheta_2(\frac{1}{6}\tau)} + \frac{\vartheta'_4(\frac{1}{6}\tau)}{\vartheta_4(\frac{1}{6}\tau)}\right] - \frac{\kappa t}{12c} = 0.$$

A rough test shows that the solution is approximately $t = 1{\cdot}7$. On account of this, the following formula is accurate to three places of decimals with a maximum possible error of $0{\cdot}0002$.

$$\frac{cU_d}{\kappa} - d = \frac{t}{8}\left\{\left[\tanh \tfrac{1}{6}\pi t - \frac{2 \sinh \frac{1}{3}\pi t}{\cosh \pi t - \cosh \frac{1}{3}\pi t}\right] - \tfrac{2}{3}\right\} = 0.$$

Put $x = \exp(\frac{1}{3}\pi t)$; this equation becomes

$$x^4 - 6x^3 - 6x + 5 = 0.$$

The solution of this is $x = 6{\cdot}1376$, giving $t = 1{\cdot}732$ and $d = 0{\cdot}144$. This suggests that the exact solution is $t = \sqrt{3}$, and, in fact, if we put $t = \sqrt{3}$ and $d = \dfrac{\sqrt{3}}{12}$, a simple consideration shows us that $u(0, -c)$ is zero. If a diagram is drawn for the case $t = \sqrt{3}$, $d = \dfrac{\sqrt{3}}{12}$, that is $a = \sqrt{3}\,\rho$, $b = 6\rho$, $c = 3\sqrt{3}\,\rho$, where ρ is some constant, we see that the configuration is such that any two consecutive vortices on one row subtend an angle of 120° at that vortex, on the other row, which is opposite the centre of the interval between these vortices. Also, relative to the point $(0, -c)$ the vortices seem to be situated at the vortices of an infinite system of adjacent regular hexagons which fill the whole of the plane, the point $(0, -c)$ being at the centre of one of these hexagons.

If we take the point $(0, -c)$ as centre and draw concentric circles of radii $4\sqrt{3}\,\rho$, $4\sqrt{12}\,\rho$, $4\sqrt{21}\,\rho$, $4\sqrt{39}\,\rho$, etc., we find that this system of circles passes through all the

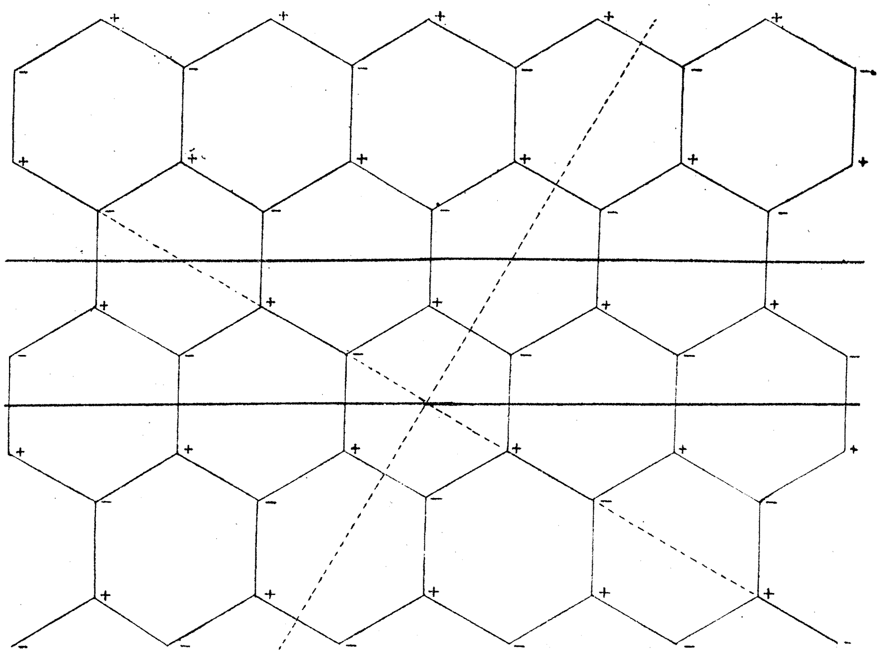

FIG. 3.

vortices in the plane. The circles pass through 6, 6, 12, 12, 18, etc., vortices respectively, the vortices being so arranged that vortices of opposite sign lie at the ends of the same diameter. The vortices are placed symmetrically and so the effect of the vortices on each of the circles is to produce zero velocity separately at $(0, -c)$. Thus the effect of the 6 vortices on the first circle is to produce zero velocity at $(0, -c)$. Similarly for the other circles—and so the total velocity at $(0, -c)$ is zero.

8.6. The above result can be shown analytically as follows: We have

$$\omega = -\mathrm{U}z - \frac{i\kappa}{2\pi}\log\frac{\vartheta_1(\mathrm{Z}-\frac{1}{12}\tau)\,\vartheta_3(\mathrm{Z}-\frac{1}{12}\tau)}{\vartheta_2(\mathrm{Z}+\frac{1}{12}\tau)\,\vartheta_4(\mathrm{Z}+\frac{1}{12}\tau)},$$

where

$$\mathrm{U} = \frac{i\kappa}{4\pi b}\left[\frac{\vartheta'_2(\frac{1}{6}\tau)}{\vartheta_2(\frac{1}{6}\tau)} + \frac{\vartheta'_4(\frac{1}{6}\tau)}{\vartheta_4(\frac{1}{6}\tau)}\right].$$

Let us change the origin to the point $(0, -c)$. Neglecting additive constants, the ω function becomes

$$-\mathrm{U}z - \frac{ik}{2\pi}\log\frac{\vartheta_1(\mathrm{Z}-\frac{1}{3}\tau)\,\vartheta_3(\mathrm{Z}-\frac{1}{3}\tau)}{\vartheta_2(\mathrm{Z}-\frac{1}{6}\tau)\,\vartheta_4(\mathrm{Z}-\frac{1}{6}\tau)}$$

$$= -\frac{i\kappa}{2\pi}\log\left\{\exp\left[i\frac{2\pi}{\kappa}\left(\frac{\kappa}{2b}-\mathrm{U}\right)z\right]\frac{\vartheta_1(\mathrm{Z}-\frac{1}{3}\tau)\,\vartheta_3(\mathrm{Z}-\frac{1}{3}\tau)}{\vartheta_1(\mathrm{Z}+\frac{1}{3}\tau)\,\vartheta_3(\mathrm{Z}+\frac{1}{3}\tau)}\right\} = -\frac{i\kappa}{2\pi}\log \mathrm{J(Z)}\ \text{(say)}.$$

This new ω function demonstrates incidentally a fact that is known already, namely, the fact that the barrier is a stream line, for if we put $Z = \xi$ where ξ is real, we have that

$$\left|\frac{\vartheta_1(\xi - \frac{1}{3}\tau)\,\vartheta_3(\xi - \frac{1}{3}\tau)}{\vartheta_1(\xi + \frac{1}{3}\tau)\,\vartheta_3(\xi + \frac{1}{3}\tau)}\right| = 1,$$

since the numerator and denominator are conjugate functions, and also

$$\left|\exp\left[i\frac{2\pi}{\kappa}\left(\frac{\kappa}{2b} - \mathrm{U}\right)\xi\right]\right| = 1 \quad \text{so that} \quad |\,\mathrm{J}(\mathrm{Z})\,| = 1.$$

8.61. The function $\vartheta_1(\mathrm{Z} - \frac{1}{3}\tau)\,\vartheta_3(\mathrm{Z} - \frac{1}{3}\tau)$ or rather $\vartheta_1\left(\frac{z}{2\omega_1} - \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\omega_1} - \frac{1}{3}\tau\right)$

has simple zeros at the positive vortices of the system. Originally the vortices were divided into groups whose periods are $2\omega_1$ and $2\omega_3$ where $\omega_3/\omega_1 = i\sqrt{3}$. Let us now divide them into groups whose periods are $2\Omega_1$ and $2\Omega_3$ where

$$2\Omega_1 = \omega_1 + \omega_3 \qquad \text{and} \qquad 2\Omega_3 = -3\omega_1 + \omega_3$$

that is

$$\tau' = \frac{2\Omega_3}{2\Omega_1} = \frac{-3\omega_1 + \omega_3}{\omega_1 + \omega_3} = \frac{-3 + i\sqrt{3}}{1 + i\sqrt{3}} = i\sqrt{3} = \tau.$$

This is equivalent to rotating the axes of reference through an angle of 60°. It is easy to see that the functions

$$\vartheta_1\left(\frac{z}{2\Omega_1} - \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1} - \frac{1}{3}\tau\right) \quad \text{and} \quad \vartheta_1\left(\frac{z}{2\omega_1} - \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\omega_1} - \frac{1}{3}\tau\right)$$

have all their zeros in common. Hence

$$\frac{\vartheta_1\left(\frac{z}{2\Omega_1} - \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1} - \frac{1}{3}\tau\right)\Big|\,\Omega_1,\ \Omega_3}{\vartheta_1\left(\frac{z}{2\omega_1} - \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\omega_1} - \frac{1}{3}\tau\right)\Big|\,\omega_1,\ \omega_3}$$

has no zeros or poles in the whole of the z plane. This is also true of the function

$$\frac{\vartheta_1\left(\frac{z}{2\Omega_1} + \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1} + \frac{1}{3}\tau\right)\Big|\,\Omega_1,\ \Omega_3}{\vartheta_1\left(\frac{z}{2\omega_1} + \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\omega_1} + \frac{1}{3}\tau\right)\Big|\,\omega_1,\ \omega_3}.$$

Now let

$$H(z) = \frac{\vartheta_1\left(\frac{z}{2\Omega_1} - \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1} - \frac{1}{3}\tau\right)\Big| \Omega_1, \Omega_3}{\vartheta_1\left(\frac{z}{2\Omega_1} + \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1} + \frac{1}{3}\tau\right)\Big| \Omega_1, \Omega_3} \div \frac{\vartheta_1\left(\frac{z}{2\omega_1} - \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\omega_1} - \frac{1}{3}\tau\right)\Big| \omega_1, \omega_3}{\vartheta_3\left(\frac{z}{2\omega_1} + \frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\omega_1} + \frac{1}{3}\tau\right)\Big| \omega_1, \omega_3}. \quad \ldots \quad (7)$$

From the preceding considerations we see immediately that $H(z)$ has neither zeros nor poles in the whole of the z plane. Also

$$H(z + 2\Omega_1) = \exp(-\tfrac{4}{3}i\pi\tau)\, H(z),$$

and

$$H(z + 2\,\Omega_3) = \exp(\tfrac{4}{3}i\pi\tau)\, H(z).$$

Therefore if we put

$$H_1(z) = \exp(\alpha z)\, H^3(z)$$

we get

$$H_1(z + 2\Omega_1) = \exp(2\alpha\Omega_1 - 4i\pi\tau)\, H_1(z),$$

and

$$H_1(z + 2\Omega_3) = \exp(2\alpha\Omega_3 + 4i\pi\tau)\, H_1(z).$$

Hence if $\alpha = 2i\pi(1 + i\sqrt{3})/\omega_1$ we have

$$H_1(z) = H_1(z + 2\Omega_1) = H_1(z + 2\Omega_3).$$

That is $H_1(z)$ is a doubly periodic function and since we know that it has neither zeros nor poles, it must be a constant. Putting $z = 0$ we see that $H_1(0) = 1$ so that $H_1(z) = 1$. Differentiating this logarithmically and putting $z = 0$, we get

$$\tfrac{2}{3}\frac{i\pi}{\omega_1}(1 + i\sqrt{3}) + \left(\frac{1}{\omega_1} - \frac{1}{\Omega_1}\right)\left(\frac{\vartheta_1'(\frac{1}{3}\tau)}{\vartheta_1(\frac{1}{3}\tau)} + \frac{\vartheta_3'(\frac{1}{3}\tau)}{\vartheta_3(\frac{1}{3}\tau)}\right) = 0,$$

that is

$$\frac{\vartheta_1'(\frac{1}{3}\tau)}{\vartheta_1(\frac{1}{3}\tau)} + \frac{\vartheta_3'(\frac{1}{3}\tau)}{\vartheta_3(\frac{1}{3}\tau)} + \frac{4i\pi}{3} = 0.$$

Put $\frac{1}{3}\tau = \frac{1}{2}\tau - \frac{1}{6}\tau$ in this expression and we get

$$\frac{\vartheta_2'(\frac{1}{6}\tau)}{\vartheta_2(\frac{1}{6}\tau)} + \frac{\vartheta_4'(\frac{1}{6}\tau)}{\vartheta_4(\frac{1}{6}\tau)} + \frac{2i\pi}{3} = 0,$$

that is

$$cUd/\kappa - d \equiv 0 \quad \text{when} \quad \tau = i\sqrt{3},$$

that is, the velocity at $(0, -c)$ is zero.

8.7. It is obvious from fig. 3 that the line $z = \eta\exp(\frac{1}{3}i\pi)$, where η is a real constant, is a stream line, for every vortex possesses an image vortex with respect to this line—and this can be shown analytically as follows. We have

$$\omega = -\frac{i\kappa}{2\pi}\log\left\{\exp\left[i\frac{2\pi}{\kappa}\left(\frac{\kappa}{2b}-\mathrm{U}\right)z\right]\frac{\vartheta_1\left(\frac{z}{2\omega_1}-\frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\omega_1}-\frac{1}{3}\tau\right)}{\vartheta_1\left(\frac{z}{2\omega_1}+\frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\omega_1}+\frac{1}{3}\tau\right)}\right\}.$$

By putting $\mathrm{H}(z) = \exp(-\alpha z)$ and making use of equation (7), this becomes

$$-\frac{i\kappa}{2\pi}\log\left\{\exp\left[i\frac{2\pi}{\kappa}\left(\frac{\kappa}{6b}-\mathrm{U}\right)z - i\frac{2\kappa}{3b}(1-i\sqrt{3})\,z\right]\frac{\vartheta_1\left(\frac{z}{2\Omega_1}-\frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1}-\frac{1}{3}\tau\right)}{\vartheta_1\left(\frac{z}{2\Omega_1}+\frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1}+\frac{1}{3}\tau\right)}\right\}.$$

But since $\mathrm{U} = \kappa/6b$, this is

$$-\frac{i\kappa}{2\pi}\log\left\{\exp\left[-i\frac{2\kappa}{3b}(1-i\sqrt{3})\,z\right]\frac{\vartheta_1\left(\frac{z}{2\Omega_1}-\frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1}-\frac{1}{3}\tau\right)}{\vartheta_1\left(\frac{z}{2\Omega_1}+\frac{1}{3}\tau\right)\vartheta_3\left(\frac{z}{2\Omega_1}+\frac{1}{3}\tau\right)}\right\}$$

$$= -\frac{i\kappa}{2\pi}\log \mathrm{K}(z) \quad \text{(say)}.$$

If $z = \eta\exp(i\pi/3)$ then

$$\frac{z}{2\Omega_1} = \frac{\eta\exp(i\pi/3)}{\omega_1(1+i\sqrt{3})} = \frac{\eta}{2\omega_1} = \text{purely real quantity.}$$

Then

$$\left|\frac{\vartheta_1\left(\frac{\eta}{2\omega_1}-\frac{1}{3}\tau\right)\vartheta_3\left(\frac{\eta}{2\omega_1}-\frac{1}{3}\tau\right)}{\vartheta_1\left(\frac{\eta}{2\omega_1}+\frac{1}{3}\tau\right)\vartheta_3\left(\frac{\eta}{2\omega_1}+\frac{1}{3}\tau\right)}\right| = 1,$$

since numerator and denominator are conjugate functions. Also

$$\left|\exp\left[-i\frac{2\kappa}{3b}(1-i\sqrt{3})\,\eta\exp(i\pi/3)\right]\right| = \left|\exp\left[-i\frac{4\kappa}{3b}\eta\right]\right| = 1,$$

thus giving $|\mathrm{K}(z)| = 1$. This we know is the condition that the line under consideration is a stream line. Hence $z = \eta\exp(i\pi/3)$ is a stream line of the system.

9. *Different Types of Relative Stream Lines.*

9.1. Let us consider the variation of S (X, Y) along the line X = 0.

$$S(0, Y) = -\exp\left(\frac{8\pi b U Y}{\kappa}\right)\frac{\vartheta_1^2(i\overline{Y-d})\,\vartheta_3^2(i\overline{Y-d})}{\vartheta_2^2(i\overline{Y+d})\,\vartheta_4^2(i\overline{Y+d})}.$$

We note that if there is a point of zero velocity on the line X = 0 then $\partial S(0, Y)/\partial y = 0$ here, that is S (0, Y) has a maximum at this point if it is plotted as a function of Y for this particular value of d. There cannot be a minimum at this point because the occurrence of a minimum would necessitate the occurrence of a maximum at some other point on this line, since S (0, Y) is always positive and S (0, d) always zero. This is impossible since it would mean that there would be two points of zero velocity on a line on which, we know, there cannot be more than one. If there is no point of zero velocity on X = 0, then S (0, Y) attains its upper bound in the range $d < Y \leq \frac{1}{4}t$ at $Y = \frac{1}{4}t$, and its upper bound in the range $-\frac{1}{4}t \leq Y < d$ at $Y = -\frac{1}{4}t$.

9.2. It is evident that when $u(0, -c)$ is positive, there must be a point of zero velocity on the line $y = -c$ for $u(b, -c)$ is always negative, and $u(x, -c)$ is a continuous function. Hence there is no point of zero velocity on the line X = 0 when $u(0, -c)$ is positive.

9.3. If the stream line S (X, Y) = 1 cuts the line X = 0 at a point above Y = d, then $S(X, \frac{1}{4}t) > 1$ on account of paragraphs 8.4 and 9.1. But

$$[S(X, \tfrac{1}{4}t)]_{d=0} = 1 \quad \text{and} \quad [S(X, \tfrac{1}{4}t)]_{d=\frac{1}{4}t} = 0.$$

Hence if the stream line S (X, Y) = 1 is to cut the line X = 0 at a point about Y = d, then

$$\left[\frac{\partial}{\partial d} S(X, \tfrac{1}{4}t)\right]_{d=0} > 0,$$

that is

$$-\left[t\left\{\frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}\right\} + 4\pi\right] > 0.$$

From equation (4), we see that this is only possible if $t > 1/0 \cdot 7096$, that is $t > 1 \cdot 4093$.

9.4. The diagrams 4 (a)–(h) are obtained from these considerations, and when the shapes of the stream lines for the case $t = 2 \cdot 060$ are investigated, the conclusions arrived at above and embodied in the diagrams are verified. This value of t is chosen because $t = 2 \cdot 060$ corresponds to $\alpha = 9°$, one of the values for which the ϑ functions of the real variable are tabulated in JAHNKE and EMDE's 'Functionentafeln.' The parameter in these tables is $\alpha = \sin^{-1} k = \sin^{-1}(\vartheta_2^2(0)/\vartheta_3^2(0))$. U_d is first plotted and from it is obtained the graph of $u(0, -c)$. A more accurate determination shows that the zeros of $u(0, -c)$ occur at $d = 0 \cdot 059$ and at $d = 0 \cdot 297$. These values of d give rise to diagrams 4 (c) and 4 (g) respectively. The limiting case 4 (e) corresponds to the maximum of S (0, Y) occurring at the point where S (0, Y) = 1. This is obtained by determining the maxima of S (0, Y) for various values of d, and then interpolating. Fig. 4 (e) corresponds to this case — $d = 0 \cdot 185$. The point of zero velocity occurs at $Y = -0 \cdot 311$ and, of course, $S(0, -0 \cdot 311) = 1$. The other diagrams are obtained by giving suitable values to d, and the calculations are made from equation (1). The appropriate values of S (X, Y), Y and d are here inserted and the resulting equation treated as a quadratic in $\vartheta_1^2(X)/\vartheta_3^2(X)$, so that the value of X can be obtained by interpolation. The diagrams for the case $t = 2 \cdot 060$ are drawn to scale and are reproduced below. The tables for this case, and, in addition, sequences of diagrams corresponding to other values of t are also given.

TABLE I.—$\alpha = 9°$, *i.e.*, $q = 0 \cdot 001548$, *i.e.*, $\tau = 2 \cdot 060i$.

v.	$\vartheta_1(v)$.	$\vartheta'_1(v)$.	$\vartheta_2(v)$.	$\vartheta'_2(v)$.	$\vartheta_3(v)$.	$\vartheta'_3(v)$.	$\vartheta_4(v)$.	$\vartheta'_4(v)$.
0·00	0·0000	1·2463	0·3967	—0·0000	1·003	—0·0000	0·9970	0·0000
0·05	0·0627	1·2309	0·3918	0·1970	1·003	0·0060	0·9970	0·0060
0·10	0·1226	1·1853	0·3773	0·3852	1·003	0·0114	0·9975	0·0114
0·15	0·1801	1·1106	0·3535	0·5658	1·002	0·0157	0·9982	0·0157
0·20	0·2382	1·0085	0·3210	0·7483	1·001	0·0185	0·9991	0·0185
0·25	0·2805	0·8812	0·2805	0·8812	1·000	0·0195	1·000	0·0195
0·30	0·3210	0·7483	0·2382	1·0085	0·9991	0·0185	1·001	0·0185
0·35	0·3535	0·5658	0·1801	1·1106	0·9982	0·0157	1·002	0·0157
0·40	0·3773	0·3852	0·1226	1·1853	0·9975	0·0114	1·003	0·0114
0·45	0·3918	0·1970	0·0627	1·2309	0·9970	0·0060	1·003	0·0060
0·50	0·3967	0·0000	0·0000	1·2463	0·9970	0·0000	1·003	0·0000
i0·05	i0·0627	1·2617	0·4016	—i0·1970	1·0033	—i0·0062	0·9967	i0·0062
0·10	0·1267	1·3082	0·4164	0·3980	1·0037	0·0130	0·9963	0·0130
0·15	0·1939	1·3870	0·4415	0·6092	1·0046	0·0212	0·9954	0·0212
0·20	0·2659	1·5001	0·4775	0·8353	1·0059	0·0314	0·9941	0·0314
0·25	0·3447	1·6509	0·5255	1·0829	1·0078	0·0448	0·9922	0·0448
0·30	0·4316	1·8413	0·5861	1·3559	1·0101	0·0625	0·9899	0·0625
0·35	0·5293	2·0778	0·6614	1·6628	1·0141	0·0865	0·9859	0·0865
0·40	0·6400	2·3653	0·7530	2·0109	1·0192	0·1191	0·9808	0·1191
0·45	0·7668	2·7112	0·8631	2·4093	1·0263	0·1636	0·9737	0·1636
0·50	0·9132	3·1265	0·9954	2·8695	1·0359	0·2246	0·9641	0·2246
0·55	1·0805	3·6141	1·1507	3·3954	1·0491	0·3076	0·9509	0·3076
0·60	1·2753	4·1934	1·3353	4·0080	1·0672	0·4212	0·9328	0·4212
0·65	1·5013	4·8758	1·5529	4·7193	1·0920	0·5768	0·9080	0·5768
0·70	1·7647	5·6784	1·8089	5·5484	1·1259	0·7896	0·8741	0·7896
0·75	2·0739	6·6284	2·1121	6·5223	1·1725	1·0825	0·8275	1·0825
0·80	2·4315	7·7327	2·4650	7·6501	1·2361	1·4814	0·7639	1·4814
0·85	2·8496	9·0277	2·8793	8·9702	1·3232	2·0277	0·6768	2·0277
0·90	3·3380	10·5432	3·3652	10·5155	1·4424	2·7758	0·5576	2·7758
0·95	3·9079	12·3147	3·9345	12·3229	1·6056	3·7998	0·3944	3·7998
1·00	4·5771	14·3916	4·6045	14·4532	1·8300	5·2076	0·1700	5·2076

TABLE II.

d.	0·00	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·40	0·45	0·50	0·515
$\frac{b}{\kappa}U_d$	0·0000	0·0750	0·1367	0·1791	0·2029	0·2109	0·2029	0·1722	0·0926	−0·1475	−2·4377	$-\infty$
$\frac{b}{\kappa}u(0, -\frac{1}{4}t)$	0·086	0·011	−0·049	−0·073	−0·071	−0·043	0·009	0·094	0·237	0·544	7·414	$+\infty$

$u(0, -\frac{1}{4}t)$ is zero when $d = 0·059$ or $0·297$.

TABLE III.—$\tau = 2·06i$.

$d = 0·050$.

$U = 0·0750\ \kappa/b$, $4\pi\,(Uc/\kappa - d) = 0·343$, $e^{0·343} = 1·409$, $e^{-0·343} = 0·710$.

S (X, Y).	Y.	0·000.	0·100.	0·200.	0·300.	0·400.	0·500.
1·409	X	0·278	0·320	0·333	0·370	0·410	0·424
1·000		0·250	0·263	0·272	0·257	0·163	—
0·710		0·222	0·229	0·219	0·158	—	—

$d = 0·059$.

$U = 0·0890\ \kappa/b$, $4\pi\,(Uc/\kappa - d) = 0·411$, $e^{0·411} = 1·507$, $e^{-0·411} = 0·664$.

S (X, Y).	Y.	0·000.	0·100.	0·200.	0·300.	0·400.	0·500.
1·507	X	0·287	0·307	0·345	0·424	0·456	0·500
1·195		0·268	0·282	0·302	0·327	0·268	—
1·000		0·250	0·263	0·272	0·272	0·156	—
0·837		0·234	0·248	0·244	0·220	—	—
0·664		0·214	0·221	0·208	0·160	—	—

$d = 0·100$.

$U = 0·1367\ \kappa/b$, $4\pi\,(Uc/\kappa - d) = 0·513$, $e^{0·513} = 1·670$, $e^{-0·513} = 0·599$.

In the curve S (X, Y) = 1·447, X = 0·500 when Y = 0·345.

S (X, Y).	Y.	0·000.	0·100.	0·200.	0·300.	0·400.	0·500.
1·447	X	0·287	0·325	0·328	0·456	0·385	—
1·000		0·250	0·283	0·303	0·291	0·174	—
0·691		0·213	0·240	0·244	0·196	—	—

Table III—*continued.*

$d = 0 \cdot 185.$

$U = 0 \cdot 1980\ \kappa/b$, $4\pi\,(Uc/\kappa - d) = 0 \cdot 239$, $e^{0 \cdot 239} = 1 \cdot 270$, $e^{-0 \cdot 239} = 0 \cdot 788$.

In the curve $S(X, Y) = 1 \cdot 000$, $X = 0 \cdot 500$ when $Y = 0 \cdot 311$, and $X = 0 \cdot 000$ when $Y = 0 \cdot 480$.

S (X, Y).	Y.	0·000.	0·015.	0·115.	0·215.	0·315.	0·415.
1·000	X	0·250	0·261	0·340	0·424	0·486	0·373

$d = 0 \cdot 250.$

$U = 0 \cdot 2109\ \kappa/b$, $4\pi\,(Uc/\kappa - d) = 0 \cdot 412$, $e^{-0 \cdot 412} = 0 \cdot 662$, $e^{0 \cdot 412} = 1 \cdot 510$.

In the curve $S(X, Y) = 0 \cdot 593$, $X = 0 \cdot 500$ when $Y = 0 \cdot 364$.

S (X, Y).	Y.	0·000.	0·100.	0·200.	0·300.	0·400.	0·500.
0·593 1·000	X	0·133 0·250	0·263 0·398	0·358 —	0·442 —	0·456 —	0·155 —

$d = 0 \cdot 297.$

$U = 0 \cdot 2040\ \kappa/b$, $4\pi\,(Uc/\kappa - d) = -1 \cdot 093$, $e^{-1 \cdot 093} = 0 \cdot 335$, $e^{1 \cdot 093} = 2 \cdot 983$.

S (X, Y).	Y.	0·000.	0·100.	0·200.	0·300.	0·400.	0·500.
0·335 1·000	X	— 0·250	0·150 —	0·280 —	0·365 —	0·450 —	0·478 —

$d = 0 \cdot 400.$

$U = 0 \cdot 0926\ \kappa/b$, $4\pi\,(Uc/\kappa - d) = -3 \cdot 828$, $e^{-3 \cdot 828} = 0 \cdot 02176$.

S (X, Y).	Y.	0·000.	0·100.	0·200.	0·300.	0·400.	0·500.
1·000 0·02176	X	0·250 —	— —	— —	— 0·094	— 0·165	— 0·194

TABLE IV.—To determine d corresponding to fig. 4 (e).

d	0·10	0·15	0·20	0·25
Maximum S (0, Y) ...	0·691	0·833	1·132	1·687
Y	—0·345	—0·310	—0·317	—0·364

S (0, Y) has a maximum value of 1 when $d = 0\cdot185$, the maximum occurring at $\mathrm{Y} = -0\cdot311$.

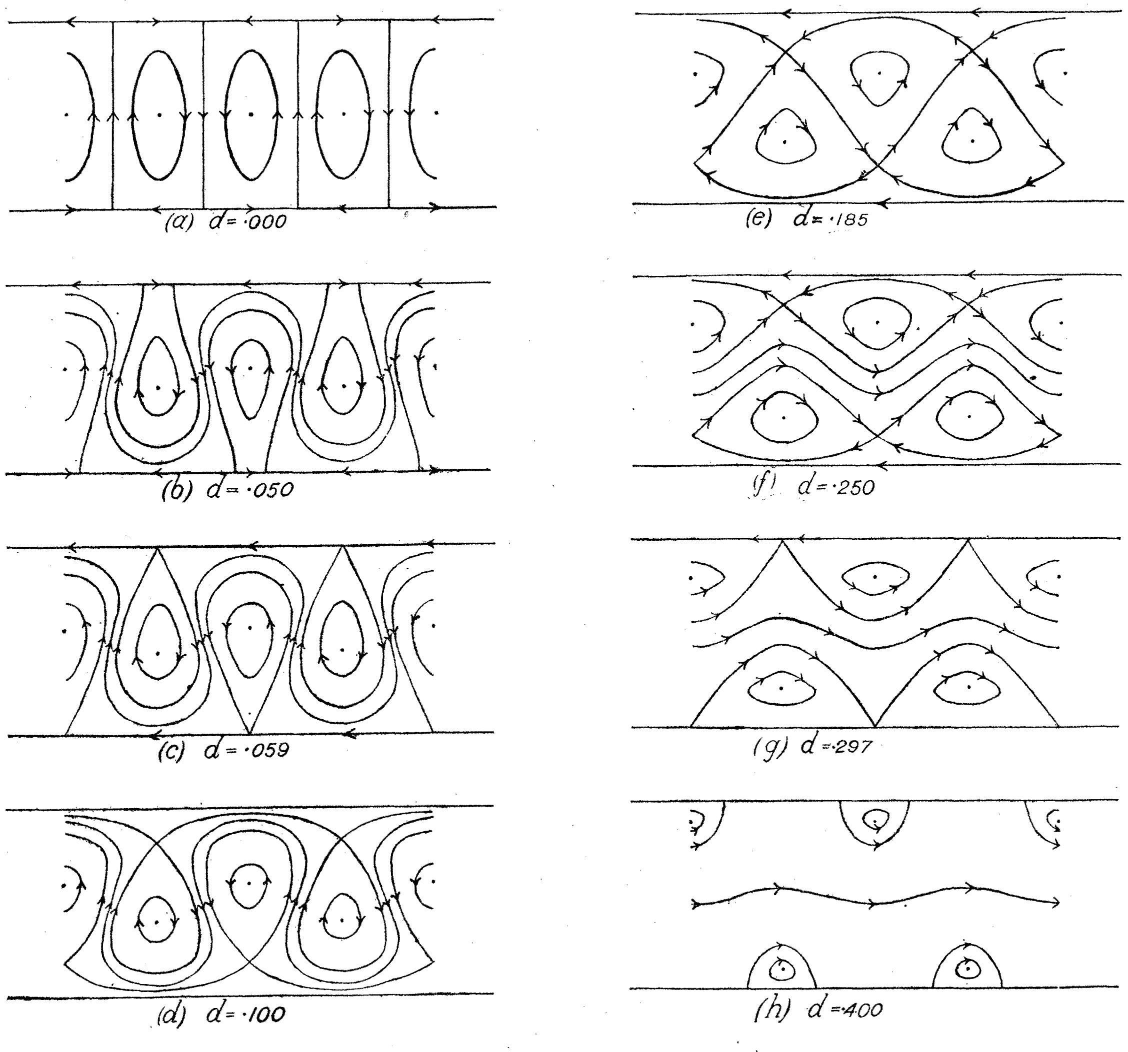

FIG. 4. $t = 2\cdot060$.

9.5. The previous sequence of diagrams is only possible when $t > 1\cdot732$. When $t = 1\cdot732$, that is $b/c = 1\cdot155$, the sequence of diagrams is as in fig. 5 (see paragraphs 8.5–8.7).

9.6. When $1\cdot732 > t > 1\cdot409$, that is $1\cdot155 < b/c < 1\cdot419$, the diagrams are as in fig. 6.

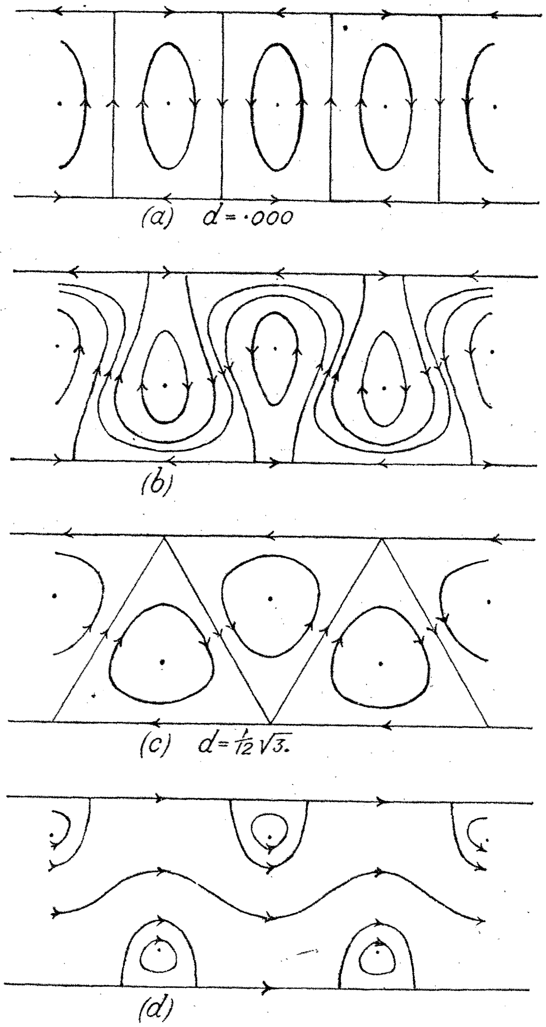

FIG. 5. $t = \sqrt{3}$.

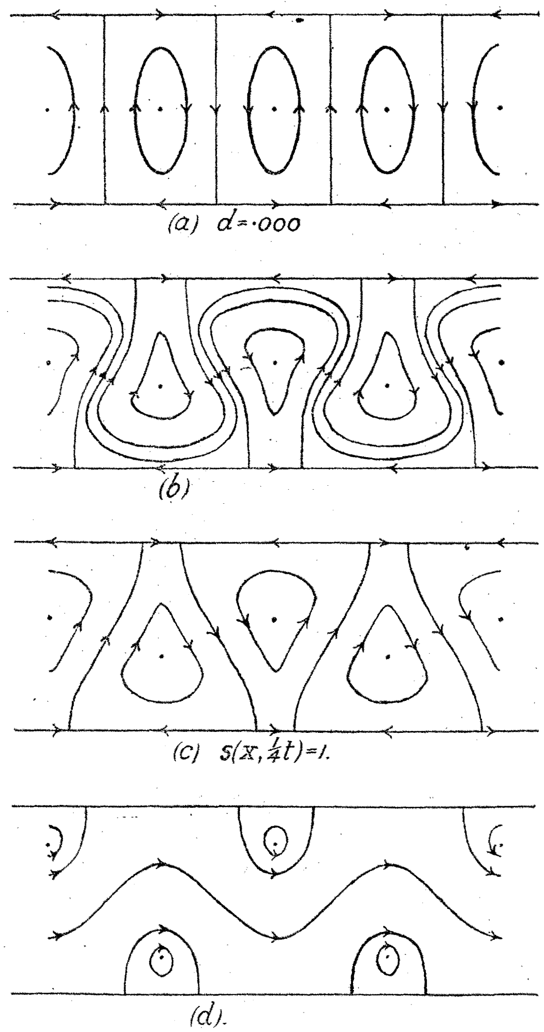

FIG. 6.

9.7. When $t \leq 1{\cdot}409$, that is $1{\cdot}419 < b/c$ the diagrams are as follows :—

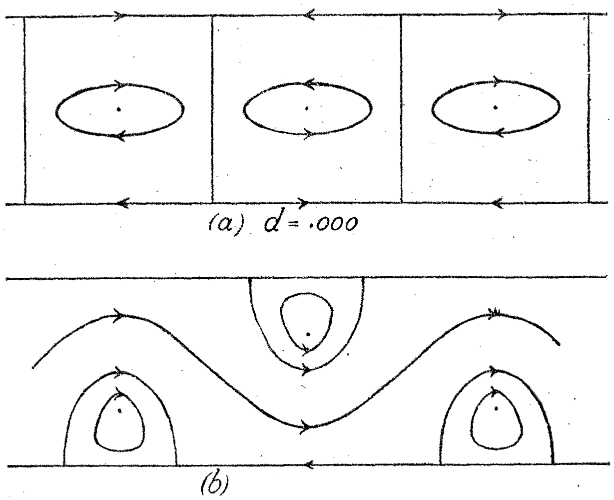

FIG. 7.

It is shown later that this final system is the only one that is stable for all values of d, for the range $t \lesssim 1 \cdot 409$ is identical with the range $\mu\, (\equiv \pi/t) \gtrsim 2 \cdot 229$.

10. *Stability.*

10.1. Let the vortices initially on the line $y = a$ be subject to a slight displacement so that the co-ordinates of the p'th vortex are $(2pb + x_p,\ a + y_p)$ where we consider the vortex initially at the point $(0, a)$ to correspond to $p = 0$. Let the vortices on the line $y = -a$ be subject to a slight displacement so that the co-ordinates of the q'th vortex are $((2q - 1)\, b + x'_q,\ -a + y'_q)$, and the vortex initially at $(b, -a)$ is considered to correspond to $q = 1$. If a vortex be subjected to a slight displacement, the infinite line of image vortices to which it gives rise is displaced in such a manner that all vortices suffer the same x displacement, and the positive and negative vortices suffer y displacements of equal magnitude and opposite sign.

10.2. Let u_0 and v_0 be the components of velocity of the vortex at $(0, a)$ due to its own image vortices. In the undisturbed state we find

$$u_0 = -\frac{\kappa}{8c} \tan \frac{\pi a}{2c}\ ; \qquad v_0 = 0.$$

If the vortex is now disturbed and takes the position $(x_0,\ a + y_0)$ we get

$$u_0 + \delta u_0 = -\frac{\kappa}{8c} \tan \frac{\pi a}{2c} - y_0 \frac{\kappa\pi}{16c^2} \sec^2 \frac{\pi a}{2c}\ ; \qquad v_0 + \delta v_0 = 0.$$

10.21. Let us now consider the velocity of the vortex at $(0, a)$ due to the vortices on the line $x = 2pb$. Let u_p and v_p be the components of this velocity. We have

$$\omega = -\frac{i\kappa}{2\pi} \log \frac{\sinh (z - 2pb - ia)\, \pi/4c}{\cosh (z - 2pb + ia)\, \pi/4c}.$$

If $u\,(x, y)$ and $v\,(x, y)$ are the components of velocity at a point (x, y) due to this system, we have

$$u\,(x, y) = \frac{\partial \psi}{\partial y} = -\frac{\kappa}{8c} \left\{ \frac{\sin [\pi\,(y - a)/2c]}{\cosh [\pi\,(x - 2pb)/2c] - \cos [\pi\,(y - a)/2c]} + \frac{\sin [\pi\,(y + a)/2c]}{\cosh [\pi\,(x - 2pb)/2c] + \cos [\pi\,(y + a)/2c]} \right\},$$

$$v\,(x, y) = -\frac{\partial \psi}{\partial x} = \frac{\kappa}{8c} \left\{ \frac{\sinh [\pi\,(x - 2pb)/2c]}{\cosh [\pi\,(x - 2pb)/2c] - \cos [\pi\,(y - a)/2c]} - \frac{\sinh [\pi\,(x - 2pb)/2c]}{\cosh [\pi\,(x - 2pb)/2c] + \cos [\pi\,(y + a)/2c]} \right\}.$$

If the vortex now suffers a displacement so that its co-ordinates are $(2pb + x_p,\ a + \delta a)$, that is $(2pb + x_p,\ a + y_p)$, we get by taking a first order approximation to the velocity at the point $(x + \delta x,\ y + \delta y)$

$$u(x+\delta x, y+\delta y) = \left[\frac{\partial \psi}{\partial y}\right]_{x,y,a} + (\delta x - x_p)\left[\frac{\partial^2 \psi}{\partial x\,\partial y}\right]_{x,y,a} + \delta y\left[\frac{\partial^2 \psi}{\partial y^2}\right]_{x,y,a} + y_p\left[\frac{\partial^2 \psi}{\partial a\,\partial y}\right]_{x,y,a},$$

$$v(x+\delta x, y+\delta y) = -\left[\frac{\partial \psi}{\partial x}\right]_{x,y,a} - (\delta x - x_p)\left[\frac{\partial^2 \psi}{\partial x^2}\right]_{x,y,a} - \delta y\left[\frac{\partial^2 \psi}{\partial x\,\partial y}\right]_{x,y,a} - y_p\left[\frac{\partial^2 \psi}{\partial a\,\partial x}\right]_{x,y,a}$$

If we evaluate these quantities, put $x = 0$, $y = a$, $\delta x = x_0$, $\delta y = y_0$, $u(0, a) = u_p$, and then sum with respect to p, we get

$$\begin{aligned}\Sigma'(u_p + \delta u_p) = &-\frac{\kappa}{8c}\Sigma'\frac{\sin 2\nu}{\cosh 2p\mu + \cos 2\nu} \\ &+\frac{\kappa\pi}{16c^2}\Sigma' x_p\frac{\sinh 2p\mu \sin 2\nu}{(\cosh 2p\mu + \cos 2\nu)^2} \\ &-\frac{\kappa\pi}{16c^2}\Sigma' y_0\left\{\frac{\cos 2\nu \cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} + \frac{1}{\cosh 2p\mu - 1}\right\} \\ &-\frac{\kappa\pi}{16c^2}\Sigma' y_p\left\{\frac{\cos 2\nu \cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} - \frac{1}{\cosh 2p\mu - 1}\right\}.\end{aligned}$$

Also

$$\begin{aligned}\Sigma'(v_p + \delta v_p) = &-\frac{\kappa\pi}{16c^2}\Sigma'(x_0 - x_p)\left\{\frac{\cos 2\nu \cosh 2p\mu + 1}{(\cosh 2\rho\mu + \cos 2v)^2} + \frac{1}{\cosh 2p\mu - 1}\right\} \\ &+\frac{\kappa\pi}{16c^2}\Sigma' y_p\frac{\sinh 2p\mu \sin 2\nu}{(\cosh 2p\mu + \cos 2\nu)^2},\end{aligned}$$

where $\mu = \dfrac{\pi b}{2c}$, and $\nu = \dfrac{\pi a}{2c}$; and where the sign Σ' denotes summation over all positive and negative integral values of p, excluding $p = 0$. We see that $0 \leq \mu \leq \infty$, $0 \leq \nu \leq \pi/2$ and that $\Sigma' v_p = 0$.

10.22. Let us now consider the velocity of the vortex at $(0, a)$ due to the vortices on the line $x = (2q - 1)b$. Let u'_q and v'_q be the components due to this line of vortices. We have

$$\omega = \frac{i\kappa}{2\pi}\log\frac{\sinh(z - (2q-1)b + ia)\pi/4c}{\cosh(z - (2q-1)b - ia)\pi/4c}.$$

If the vortex suffers a displacement so that its co-ordinates are $((2q-1)\,b + x'_q,\ -(a+\delta a))$, that is $((2q-1)\,b + x'_q,\ -a + y'_q)$, we get in a similar way:

$$\Sigma\,(u'_q + \delta u'_q) = \frac{\kappa}{8c}\Sigma\,\frac{\sin 2\nu}{\cosh\,(2q-1)\,\mu - \cos 2\nu}$$

$$-\frac{\kappa\pi}{16c^2}\,\Sigma\,x'_q\,\frac{\sin 2\nu\,\sinh\,(2q-1)\,\mu}{(\cosh\,(2q-1)\,\mu - \cos 2\nu)^2}$$

$$+\frac{\kappa\pi}{16c^2}\,\Sigma\,y_0\left\{\frac{\cos 2\nu\,\cosh\,(2q-1)\,\mu - 1}{(\cosh\,(2q-1)\,\mu - \cos 2\nu)^2} + \frac{1}{\cosh\,(2q-1)\,\mu + 1}\right\}$$

$$+\frac{\kappa\pi}{16c^2}\,\Sigma\,y'_q\left\{\frac{\cos 2\nu\,\cosh\,(2q-1)\,\mu - 1}{(\cosh\,(2q-1)\,\mu - \cos 2\nu)^2} - \frac{1}{\cosh\,(2q-1)\,\mu + 1}\right\},$$

$$\Sigma\delta v'_q = \frac{\kappa\pi}{16c^2}\,\Sigma\,(x_0 - x'_q)\left\{\frac{\cos 2\nu\,\cosh\,(2q-1)\,\mu - 1}{(\cosh\,(2q-1)\,\mu - \cos 2\nu)^2} + \frac{1}{\cosh\,(2q-1)\,\mu + 1}\right\}$$

$$+\frac{\kappa\pi}{16c^2}\,\Sigma\,y'_q\,\frac{\sinh\,(2q-1)\,\mu\,\sin 2\nu}{(\cosh\,(2q-1)\,\mu - \cos 2\nu)^2},$$

where the sign Σ denotes summation over all positive and negative integral values of q, including $q = 0$.

10.23. Hence, if V be the velocity of the vortex at $(0, a)$ in the undisturbed state, we have

$$\mathrm{V} + \frac{dx_0}{dt} = u_0 + \delta u_0 + \Sigma'\,(u_p + \delta u_p) + \Sigma\,(u'_q + \delta u'_q)$$

$$= u_0 + \Sigma'\,u_p + \Sigma u'_q + \Sigma'\,\delta u_p + \Sigma\,\delta u'_q$$

$$= -\frac{\kappa}{8c}\tan\nu - \frac{\kappa}{8c}\Sigma'\frac{\sin 2\nu}{\cosh 2p\mu + \cos 2\nu} + \frac{\kappa}{8c}\Sigma\frac{\sin 2\nu}{\cosh\,(2q-1)\mu - \cos 2\nu}$$

$$+\frac{\kappa\pi}{16c^2}\Sigma'\,x_p\,\frac{\sinh 2p\mu\,\sin 2\nu}{(\cosh 2p\mu + \cos 2\nu)^2}$$

$$-\frac{\kappa\pi}{16c^2}y_0\left[\sec^2\nu + \Sigma'\left\{\frac{\cos 2\nu\,\cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} + \frac{1}{\cosh 2p\mu - 1}\right\}\right.$$

$$\left. - \Sigma\left\{\frac{\cos 2\nu\,\cosh\,(2q-1)\,\mu - 1}{\cosh\,(2q-1)\,\mu - \cos 2\nu} + \frac{1}{\cosh\,(2q-1)\,\mu + 1}\right\}\right]$$

$$-\frac{\kappa\pi}{16c^2}\,\Sigma'\,y_p\left\{\frac{\cos 2\nu\,\cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} - \frac{1}{\cosh 2p\mu - 1}\right\}$$

$$-\frac{\kappa\pi}{16c^2}\,\Sigma\,x'_q\,\frac{\sinh\,(2q-1)\,\mu\,\sin 2\nu}{(\cosh\,(2q-1)\,\mu - \cos 2\nu)^2}$$

$$-\frac{\kappa\pi}{16c^2}\,\Sigma\,y'_q\left\{\frac{\cos 2\nu\,\cosh\,(2q-1)\,\mu - 1}{(\cosh\,(2q-1)\,\mu - \cos 2\nu)^2} - \frac{1}{\cosh\,(2q-1)\,\mu + 1}\right\}.$$

In this expression the terms independent of the disturbance are

$$-\frac{\kappa}{8c}\tan\nu-\frac{\kappa}{8c}\Sigma'\frac{\sin 2\nu}{\cosh 2p\mu+\cos 2\nu}+\frac{\kappa}{8c}\Sigma\frac{\sin 2\nu}{\cosh(2q-1)\mu-\cos 2\nu}\quad . \quad . \qquad (8)$$

This and similar expressions can be simplified by means of the following formulæ

$$\frac{\tau}{\pi}\frac{\vartheta'_1(\nu\tau/\pi\mid\tau)}{\vartheta_1(\nu\tau/\pi\mid\tau)}=\cot\nu+\frac{2\nu}{\mu}+2\sum_1^\infty\frac{\sin 2\nu}{\cosh 2p\mu-\cos 2\nu},$$

$$\frac{\tau}{\pi}\frac{\vartheta'_2(\nu\tau/\pi\mid\tau)}{\vartheta_2(\nu\tau/\pi\mid\tau)}=\frac{2\nu}{\mu}+2\sum_1^\infty\frac{\sin 2\nu}{\cosh(2p-1)\mu-\cos 2\nu},$$

$$\frac{\tau}{\pi}\frac{\vartheta'_3(\nu\tau/\pi\mid\tau)}{\vartheta_3(\nu\tau/\pi\mid\tau)}=\frac{2\nu}{\mu}+2\sum_1^\infty\frac{-\sin 2\nu}{\cosh(2p-1)\mu+\cos 2\nu}.$$

$$\frac{\tau}{\pi}\frac{\vartheta'_4(\nu\tau/\pi\mid\tau)}{\vartheta_4(\nu\tau/\pi\mid\tau)}=-\tan\nu+\frac{2\nu}{\mu}+2\sum_1^\infty\frac{-\sin 2\nu}{\cosh 2p\mu+\cos 2\nu},$$

where μ and ν are real and where $\tau=\pi i/\mu$. Hence the expression (8) equals

$$\frac{\kappa}{8\pi c}\left[-8\pi d+i\frac{2c}{b}\left\{\frac{\vartheta'_2(2id)}{\vartheta_2(2id)}+\frac{\vartheta'_4(2id)}{\vartheta_4(2id)}\right\}\right]=\mathrm{U}_d-\frac{\kappa d}{c}.$$

But, in summing the velocities contributed by the various lines of vortices, we have been using the ω function

$$-\frac{\kappa d}{c}z-\frac{i\kappa}{2\pi}\log f(\mathrm{Z})$$

as shown in paragraph 2.23. Hence V equals $\mathrm{U}-\kappa d/c$, and the terms independent of the disturbance cancel. Similarly we get

$$\frac{dy_0}{dt}=\Sigma' v_p+\Sigma v'_q$$

$$=-\frac{\kappa\pi}{16c^2}x_0\left[\Sigma'\left\{\frac{\cos 2\nu\cosh 2p\mu+1}{(\cosh 2p\mu+\cos 2\nu)^2}+\frac{1}{\cosh 2p\mu-1}\right\}\right.$$

$$\left.-\Sigma\left\{\frac{\cos 2\nu\cosh(2q-1)\mu-1}{(\cosh(2q-1)\mu-\cos 2\nu)^2}+\frac{1}{\cosh(2q-1)\mu+1}\right\}\right]$$

$$+\frac{\kappa\pi}{16c^2}\Sigma' x_p\left\{\frac{\cos 2\nu\cosh 2p\mu+1}{(\cosh 2p\mu+\cos 2\nu)^2}+\frac{1}{\cosh 2p\mu-1}\right\}$$

$$+\frac{\kappa\pi}{16c^2}\Sigma' y_p\frac{\sin 2\nu\sinh 2p\mu}{(\cosh 2p\mu+\cos 2\nu)^2}$$

$$-\frac{\kappa\pi}{16c^2}\Sigma x'_q\left\{\frac{\cos 2\nu\cosh(2q-1)\mu-1}{(\cos(2q-1)\mu-\cos 2\nu)^2}+\frac{1}{\cosh(2q-1)\mu+1}\right\}$$

$$+\frac{\kappa\pi}{16c^2}\Sigma y'_q\frac{\sin 2\nu\sinh(2q-1)\mu}{(\cosh(2q-1)\mu-\cos 2\nu)^2}.$$

11. *The Type of Disturbance.*

There are an infinite number of such equations but they are reduced to two if we put $x_p = \alpha \cos 2p\phi$, $y_p = \beta \cos 2p\phi$; $x'_q = \alpha' \cos(2q-1)\phi$, $y'_q = \beta' \cos(2q-1)\phi$. α, β, α' and β' are functions of the time and ϕ is some real constant such that $0 \leq \phi \leq \pi$. The factors $\cos 2p\phi$ and $\cos(2q-1)\phi$ introduce a periodicity depending upon position into the displacements of the vortices—the wave-length of this disturbance being $2\pi b/\phi$.

12. *The Equations of Motion.*

12.1. We get

$$\rho\dot{\alpha} = A\alpha + B\beta + A'\alpha' + B'\beta'$$
$$\rho\dot{\beta} = C\alpha + D\beta + C'\alpha' + D'\beta',$$

where $\rho = 16c^2/\kappa\pi$ and where

$$A = \Sigma' \cos 2p\phi \frac{\sinh 2p\mu \sin 2\nu}{(\cosh 2p\mu + \cos 2\nu)^2} = 0,$$

$$\begin{aligned} B = -\sec^2\nu - \Sigma' &\left\{\frac{\cos 2\nu \cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} + \frac{1}{\cosh 2p\mu - 1}\right\} \\ + \Sigma &\left\{\frac{\cos 2\nu \cosh(2q-1)\mu - 1}{(\cosh(2q-1)\mu - \cos 2\nu)^2} + \frac{1}{\cosh(2q-1)\mu + 1}\right\} \\ - \Sigma' \cos 2p\phi &\left\{\frac{\cos 2\nu \cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} - \frac{1}{\cosh 2p\mu - 1}\right\}, \end{aligned}$$

$$A' = -\Sigma \cos(2q-1)\phi \frac{\sinh(2q-1)\mu \sin 2\nu}{(\cosh(2q-1)\mu - \cos 2\nu)^2} = 0,$$

$$B' = -\Sigma \cos(2q-1)\phi \left\{\frac{\cos 2\nu \cosh(2q-1)\mu - 1}{(\cosh(2q-1)\mu - \cos 2\nu)^2} - \frac{1}{\cosh(2q-1)\mu + 1}\right\},$$

$$\begin{aligned} C = -\Sigma' &\left\{\frac{\cos 2\nu \cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} + \frac{1}{\cosh 2p\mu - 1}\right\} \\ + \Sigma &\left\{\frac{\cos 2\nu \cosh(2q-1)\mu - 1}{(\cosh(2q-1)\mu - \cos 2\nu)^2} + \frac{1}{\cosh(2q-1)\mu + 1}\right\} \\ + \Sigma' \cos 2p\phi &\left\{\frac{\cos 2\nu \cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} + \frac{1}{\cosh 2p\mu - 1}\right\}, \end{aligned}$$

$$D = \Sigma' \cos 2p\phi \left\{\frac{\sinh 2p\mu \sin 2\nu}{(\cosh 2p\mu + \cos 2\nu)^2}\right\} = 0,$$

$$C' = -\Sigma \cos(2q-1)\phi \left\{\frac{\cos 2\nu \cosh(2q-1)\mu - 1}{(\cosh(2q-1)\mu - \cos 2\nu)^2} + \frac{1}{\cosh(2q-1)\mu + 1}\right\},$$

$$D' = \Sigma \cos(2q-1)\phi \left\{\frac{\sinh(2q-1)\mu \sin 2\nu}{(\cosh(2q-1)\mu - \cos 2\nu)^2}\right\} = 0.$$

The equations therefore are

$$\left.\begin{aligned} \rho\dot{\alpha} &= B\beta + B'\beta' \\ \rho\dot{\beta} &= C\alpha + C'\alpha' \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (9)$$

To deduce the equations relating to the lower row we have merely to reverse the signs of κ and a, and to interchange accented and unaccented letters. Hence

$$\left.\begin{aligned} \rho\dot{\alpha}' &= \qquad\quad - B'\beta \qquad\quad - B\beta' \\ \rho\dot{\beta}' &= - C'\alpha \qquad\quad - C\alpha' \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (10)$$

Putting $\alpha = \alpha_0 e^{\lambda t}$, $\beta = \beta_0 e^{\lambda t}$, $\alpha' = \alpha_0' e^{\lambda t}$ and $\beta' = \beta_0' e^{\lambda t}$, we get in the usual way

$$\{\rho^2\lambda^2 - (B - B')(C + C')\}\{\rho^2\lambda^2 - (B + B')(C - C')\} = 0.$$

The solutions are therefore of two types. In the first type we have

$$\alpha = \alpha', \qquad \beta = -\beta'.$$

This solution involves exponentials $e^{\lambda t}$ where λ is given by

$$\lambda = \pm\frac{1}{\rho}\sqrt{(B - B')(C + C')}.$$

In the second type we have

$$\alpha = -\alpha', \qquad \beta = \beta'.$$

This involves exponentials $e^{\lambda t}$ where

$$\lambda = \pm\frac{1}{\rho}\sqrt{(B + B')(C - C')}.$$

For stability therefore we must have $(B - B')(C + C')$ and $(B + B')(C - C')$ either negative or zero in the range $0 \leq \phi \leq \pi$. The cases where these expressions are zero are included in the stable type for the period of the disturbance is then infinite.

12.2. We note, however, that if we put $(\pi - \phi)$ instead of ϕ, α and β remain unaltered but α' and β' change sign. This means that the first type of solution is transformed into the second. This is easily verified for if we put $(\pi - \phi)$ instead of ϕ in $(B - B')(C + C')$ we get $(B + B')(C - C')$. It is therefore only necessary to discuss the expression $(B - B')(C + C')$.

13. $\mu \to 0, \quad \nu \to 0.$

When μ and ν are both vanishingly small we get

$$2(B - B') = X - Y \qquad \text{and} \qquad 2(C + C') = X + Y,$$

where

$$X = \frac{\pi}{\mu}\tanh\frac{\pi}{\mu} + \frac{[\pi/\mu]^2}{\cosh^2[\nu\pi/\mu]} - \frac{\pi/\mu}{(1-\nu^2)^{\frac{1}{2}}}\coth[(1-\nu^2)^{\frac{1}{2}}\pi/\mu] - \frac{2\phi(\pi-\phi)}{\mu^2} - \frac{\pi}{\mu}\frac{\sinh(\pi-2\phi)/\mu}{\cosh\pi/\mu},$$

$$Y = \frac{\pi^2}{\mu^2}\frac{\cosh[2\nu\phi/\mu]}{\cosh^2[\nu\pi/\mu]} - \frac{\pi/\mu}{(1-\nu^2)^{\frac{1}{2}}}\frac{\cosh[(1-\nu^2)^{\frac{1}{2}}(\pi-2\phi)/\mu]}{\sinh[(1-\nu^2)^{\frac{1}{2}}\pi/\mu]} - \frac{2\pi\phi}{\mu^2}\frac{\cosh\nu(\pi-2\phi)/\mu}{\cosh\nu\pi/\mu}.$$

For future reference we note that

$$(\mathrm{B}-\mathrm{B}')_{\phi=0} = \left[\frac{(\pi/\mu)^2}{\cosh^2(\nu\pi/\mu)} - \frac{\pi/\mu}{(1-\nu^2)^{\frac{1}{2}}}\coth(1-\nu^2)^{\frac{1}{2}}\pi/\mu\right]. \quad . \quad . \quad . \quad (11)$$

If now we let $c \to \infty$, while a and b are finite, then by neglecting powers of μ and ν greater than the second, and putting $\nu/\mu = k$, we get

$$\underset{c\to\infty}{\mathrm{Lt}}\,(2\mathrm{B}\mu^2) = \Sigma\,\frac{(q+\frac{1}{2})^2-k^2}{[(q+\frac{1}{2})^2+k^2]^2} - \Sigma'\,\frac{1-\cos 2p\phi}{p^2}$$

$$= \frac{\pi^2}{\cosh^2 k\pi} - 2\phi(\pi-\phi) = \mathrm{B}_0 \quad \text{(say)},$$

$$\underset{c\to\infty}{\mathrm{Lt}}\,(2\mathrm{C}\mu^2) = \mathrm{B}_0,$$

$$\underset{c\to\infty}{\mathrm{Lt}}\,(2\mathrm{B}'\mu^2) = -\,\Sigma\,\frac{(q+\frac{1}{2})^2-k^2}{[(q+\frac{1}{2})^2+k^2]}\cos(2q+1)\,\phi$$

$$= \frac{2\pi\phi\cosh k(\pi-2\phi)}{\cosh k\pi} - \frac{\pi^2\cosh 2k\phi}{\cosh^2 k\pi} = \mathrm{B}_0' \quad \text{(say)},$$

$$\underset{c\to\infty}{\mathrm{Lt}}\,(2\mathrm{C}'\mu^2) = \mathrm{B}'_0.$$

This means that the two types of solution become identical and that stability depends on λ where $\lambda = \pm\sqrt{\mathrm{B}_0{}^2 - \mathrm{B}'_0{}^2}/2\rho\mu^2 = \pm\sqrt{\mathrm{B}_0{}^2 - \mathrm{B}'_0{}^2}/8\pi b^2$. When $\phi = \pi/2$, $\mathrm{B}'_0 = 0$ so that we must have $\mathrm{B}_0 = 0$ for this value of ϕ. This condition is only fulfilled when $\cosh^2 k\pi = 2$, that is when $\nu/\mu = a/b = 0\cdot281$. This result was obtained by KÁRMÁN but it was LAMB (*loc. cit.*) who gave a detailed proof that $\mathrm{B}_0{}^2 - \mathrm{B}'_0{}^2$ is always negative when $k = 0\cdot281$. This means that if μ is plotted against ν, the slope of the tangent at the origin is $3\cdot559$.

14. $\nu = 0$.

We must also consider the case where $\nu = 0$ and μ is finite. It is at once evident that $(\mathrm{C}+\mathrm{C}')_{\phi=0} = 0$ and that $\left[\frac{d}{d\phi}(\mathrm{C}+\mathrm{C}')\right]_{\phi=0} = 0$. It is necessary to know whether $(\mathrm{C}+\mathrm{C}')$ is positive or negative in the neighbourhood of $\phi = 0$. This is determined by the sign of $\left[\frac{d^2}{d\phi^2}(\mathrm{C}+\mathrm{C}')\right]_{\phi=0}$. When $\nu = 0$ we find that this is always positive so that the curve $(\mathrm{C}+\mathrm{C}')$ always has its concavity upwards in the neighbourhood of $\phi = 0$. This necessitates $(\mathrm{B}-\mathrm{B}')$ negative in the neighbourhood of $\phi = 0$. When $\phi = 0$ we get

$$(\mathrm{B}-\mathrm{B}')_{\phi=0} = 4\sum_1^\infty\left(\frac{1}{\cosh(2q-1)\mu-1} - \frac{1}{\cosh 2q\mu+1}\right) - 1$$

$$= \frac{\tau^2}{\pi^2}\left[\frac{\vartheta''_2(0)}{\vartheta_2(0)} + \frac{\vartheta''_4(0)}{\vartheta_4(0)} + \frac{4\pi i}{\tau}\right] \quad \text{where} \quad \tau = \frac{\pi i}{\mu}.$$

We get $0 \geq (B - B')_{\phi=0}$ when $\mu \leq 2 \cdot 229$. Hence when $\nu = 0$ the system cannot be stable when $\mu < 2 \cdot 229$. When $\mu = 2 \cdot 229$ we have :—

$$C + C' = 0 \cdot 8080 - 0 \cdot 8916 \cos \phi - 0 \cdot 0100 \cos 3\phi + 0 \cdot 0926 \cos 2\phi + 0 \cdot 0010 \cos 4\phi,$$

$$B - B' = -0 \cdot 1920 + 0 \cdot 1896 \cos \phi + 0 \cdot 0001 \cos 3\phi + 0 \cdot 0022 \cos 2\phi + 0 \cdot 0001 \cos 4\phi.$$

It is easy to verify that $(C + C')$ is always positive and $(B - B')$ always negative. When $\mu > 2 \cdot 229$ we find a similar state of affairs. Hence when $\nu = 0$, $\mu \geq 2 \cdot 229$ for stability.

15. *General Case. μ and ν any Value.*

15.1. Let us return to the general case. The product $(B - B')(C + C')$ may be negative in three different ways :—

(1) $(B - B')$ always positive and $(C + C')$ always negative.

(2) $(B - B')$ and $(C + C')$ always of opposite sign for the same value of ϕ, but intersecting on the ϕ axis at the same point or points.

(3) $(B - B')$ always negative and $(C + C')$ always positive.

15.2. Type (1) may be ruled out immediately because if $(C + C')$ is to be negative at $\phi = \pi$, both μ and ν must be small, and when μ and ν are small we have seen that the system is only stable when $\nu/\mu = 0 \cdot 281$—and in this case the stability is of type (2) for $(B - B')$ and $(C + C')$ intersect on the ϕ axis and change sign, at the point $\phi = \frac{1}{2}\pi$.

15.3. *Stability of Type* (3).

15.31. *Discussion of* $(C + C')$.

We know that when $\phi = 0$, $(C + C') = 0$ and $\frac{d}{d\phi}(C + C') = 0$. Hence if $(C + C')$ is to be positive it is essential that $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0}$ should be positive. This condition is a necessary one but not, as far as we know at present, a sufficient one. However, if some value is assigned to ν (other than $\nu = 0$, as this case has been considered independently) and we consider the variation in the form of the curve $(C + C')$ as μ increases from 0 to ∞, we find that $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0}$ equals $-\infty$ when $\mu = 0$, and is positive, approaching zero, when μ is $+\infty$. Hence at some point in the range, $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0} = 0$. When $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0} < 0$, the curve $(C + C')$ is convex upwards in the neighbourhood of $\phi = 0$, $(C + C')$ decreases till it attains a minimum, it then increases—cuts the axis of ϕ in some point, and remains positive for all succeeding values of ϕ. As μ increases in value, the point of section with the ϕ axis tends to and finally attains the point $\phi = 0$. For this value of μ, the curve $(C + C')$ has three zeros at the origin, that is $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0} = 0$. This is the limiting case. When μ is greater than this value, $(C + C')$ is always positive in the neighbourhood of $\phi = 0$, and from empirical determinations and from the general form of the curve we see that $(C + C')$ is positive in the range $0 \leq \phi \leq \pi$. We see therefore that the necessary and sufficient condition for $(C + C')$ to be positive is $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0} \geq 0$. The curve $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0} = 0$ has been drawn and the values of μ obtained for different values of ν give lower limits for stability.

ν.	0.	$\frac{1}{6}\pi$.	$\frac{1}{4}\pi$.	$\frac{1}{3}\pi$.	$\frac{1}{2}\pi$.
μ	0·000	1·640	1·930	2·103	2·229

15.32. *Discussion of* $(B - B')$.

The other condition for type (3) is that $(B - B')$ is always negative. Since $(B - B')$ is a continuous function it is essential that at least $(B - B')_{\phi=0} \leq 0$, so that in the neighbourhood of $\phi = 0$, the product $(B - B')(C + C')$ should be negative. As before, this condition is necessary but not, as far as we know at present, sufficient. When ν is small, *e.g.*, $\nu = 0 \cdot 01$, we find that if $(B - B')$ is to be negative, μ cannot lie between the values 0·009 and 2·228. The lower limit is obtained from equation (11), and the upper limit from the exact expression for $(B - B')_{\phi=0}$. When $\mu \leq 0 \cdot 009$, $(B - B')$ is found to be negative and the curve $(C + C')$ is found to be either negative for all values of ϕ, or to intersect the ϕ axis at some point. The system therefore cannot be stable in the range $0 \leq \mu \leq 0 \cdot 009$. When $\mu \geq 2 \cdot 228$ we find that $(B - B')$ is always negative (and from the condition for $(C + C')$ we find $(C + C')$ always positive). A similar variation in the form of $(B - B')$ occurs for other values of ν, *e.g.*, $\nu = 13°$, where the limits are $\mu = 2 \cdot 000$ and $\mu = 0 \cdot 625$. Hence if $(B - B')$ is to be negative, then $(B - B')_{\phi=0} \leq 0$. The justification for the adoption of this criterion is of an empirical nature, but it certainly defines ranges in which $(B - B')$ is always negative.

15.33. When $\nu = 0$, the two roots of the equation $(B - B')_{\phi=0} = 0$ are $\mu = 0 \cdot 000$ and $\mu = 2 \cdot 229$, and the tangents at these points are parallel to the axis of ν for $\frac{\partial}{\partial\nu}(B - B')_{\phi=0} = 0$ when $\nu = 0$. As ν increases, the two roots of the equation $(B - B')_{\phi=0} = 0$ approach and ultimately coincide. At this point the tangent to the curve is perpendicular to the ν axis. This point is determined as follows. The equation $(B - B')_{\phi=0} = 0$ can be written

$$4\sum_1^\infty \frac{\cos 2\nu \cosh(2q-1)\mu - 1}{(\cosh(2q-1)\mu - \cos 2\nu)^2} - 4\sum_1^\infty \frac{\cos 2\nu \cosh 2q\mu + 1}{(\cosh 2q\mu + \cos 2\nu)^2} = \sec^2 \nu. \quad . \quad (12)$$

The terms on the left-hand side form a rapidly convergent series whose terms are alternately positive and negative; the first term is therefore the most important term. Let the sum of the series on the left-hand side be ρ times the first term, where ρ, of course, is less than unity. Putting $\cos 2\nu = c$ and $\cosh \mu = x$, the equation becomes

$$4\frac{xc-1}{(x-c)^2}\rho = \frac{2}{1+c},$$

that is

$$x^2 - 2xc(1 + \rho + \rho c) + c^2 + 2\rho c + 2\rho = 0.$$

If the roots of this equation are equal, then

$$c^2(1 + \rho + \rho c)^2 = c^2 + 2\rho c + 2\rho,$$

that is

$$(c+1)^2[\rho c^2 + 2c - 2] = 0,$$

that is $c = \frac{1}{\rho}(\sqrt{1+2\rho} - 1)$ and the corresponding value of x is

$$\left(\frac{1}{\rho}+1\right) - \left(\frac{1}{\rho}-1\right)\sqrt{1+2\rho}.$$

As a first approximation put $\rho = 1$; this gives $c = \sqrt{3} - 1$, that is $\nu = 21° \, 28'$, and $x = 2$, that is $\mu = 1 \cdot 317$. If the terms on the left-hand side of equation (12) are evaluated with these parameters it is found that $\rho = 0 \cdot 715$. This gives a second approximation and we get $\nu = 19° \, 18'$ and $\mu = 1 \cdot 178$. The third approximation gives $\nu = 19° \, 18'$, $\mu = 1 \cdot 177$. This is the required point and our information now enables us to draw the curve $(B - B')_{\phi=0} = 0$. We find that it cuts the curve $\left[\frac{\partial^2}{\partial\phi^2}(C + C')\right]_{\phi=0} = 0$ at $\nu = 0$, $\mu = 0$, and at the point $\mu = 1 \cdot 28$, $\nu = 18° \, 45' = 0 \cdot 327$.

15.34. If $(B - B')$ is to be negative for all values of ϕ the point (μ, ν) must lie outside the domain enclosed by the curve $(B - B')_{\phi=0} = 0$ and the μ axis. If $(C + C')$ is to be positive for all values of ϕ, the point (μ, ν) must lie outside the area enclosed by the curve $\left[\frac{\partial^2}{\partial\phi^2}(C + C')\right]_{\phi=0} = 0$, the axis of ν and the line $\nu = \frac{1}{2}\pi$. Obviously then, all systems that have their parametrical point (μ, ν) in the space above both these two curves are stable. This domain is called the "Stability Area." (See fig. 9.)

15.4. *Another Form of the Criteria for Stability.*

The conditions $\left[\frac{\partial^2}{\partial \phi^2}(C+C')\right]_{\phi=0} \geq 0$ and $(B-B')_{\phi=0} \leq 0$ may be expressed in another way:

$$\left[\frac{\partial^2}{\partial \phi^2}(C+C')\right]_{\phi=0} = 2\sum_1^\infty (2p-1)^2 \left\{\frac{\cos 2\nu \cosh(2p-1)\mu - 1}{(\cosh(2p-1)\mu - \cos 2\nu)^2} + \frac{1}{\cosh(2p-1)\mu + 1}\right\}$$

$$- 2\sum_1^\infty (2p)^2 \left\{\frac{\cos 2\nu \cosh 2p\mu + 1}{(\cosh 2p\mu + \cos 2\nu)^2} + \frac{1}{\cosh 2p\mu - 1}\right\},$$

$$= 2\sum_1^\infty \frac{\partial^2}{\partial \mu^2} \log \frac{\cosh(2p-1)\mu + 1}{\cosh(2p-1)\mu - \cos 2\nu}$$

$$+ 2\sum_1^\infty \frac{\partial^2}{\partial \mu^2} \log \frac{\cosh 2p\mu - 1}{\cosh 2p\mu + \cos 2\nu},$$

$$= 2\frac{\partial^2}{\partial \mu^2} \log \prod_1^\infty \frac{\cosh(2p-1)\mu + 1}{\cosh(2p-1)\mu - \cos 2\nu} \prod_1^\infty \frac{\cosh 2p\mu - 1}{\cosh 2p\mu + \cos 2\nu}.$$

We know, however, that $e^{-\mu} = q_1$, hence

$$\left[\frac{\partial^2}{\partial \phi^2}(C+C')\right]_{\phi=0} = 2\frac{\partial^2}{\partial \mu^2} \log \frac{\vartheta_3\left(0 \,\middle|\, -\frac{1}{\tau}\right)\vartheta'_1\left(0 \,\middle|\, -\frac{1}{\tau}\right)}{\vartheta_4\left(\frac{\nu}{\pi} \,\middle|\, -\frac{1}{\tau}\right)\vartheta_2\left(\frac{\nu}{\pi} \,\middle|\, -\frac{1}{\tau}\right)}.$$

Putting $\tau_1 (\equiv it_1) = -\frac{1}{\tau}$, we get $\mu = \frac{i\pi}{\tau} = \pi t_1$. Also, by putting $\frac{\nu}{\pi} = \nu_1$ the condition $\left[\frac{\partial^2}{\partial \phi^2}(C+C')\right]_{\phi=0} \geq 0$ becomes

$$\frac{\partial^2}{\partial t_1^2} \log \frac{\vartheta_1'(0|\tau_1)\,\vartheta_3(0|\tau_1)}{\vartheta_2(\nu_1|\tau_1)\,\vartheta_4(\nu_1|\tau_1)} \geq 0. \quad \text{. . . . . (13)}$$

Similarly the condition $(B-B')_{\phi=0} \leq 0$ becomes

$$\frac{\partial^2}{\partial \nu^2} \log \vartheta_2(\nu_1|\tau_1)\,\vartheta_4(\nu_1|\tau_1) \leq 0,$$

or

$$\frac{\partial^2}{\partial \nu_1^2} \log \frac{\vartheta_1'(0|\tau_1)\,\vartheta_3(0|\tau_1)}{\vartheta_2(\nu_1|\tau_1)\,\vartheta_4(\nu_1|\tau_1)} \geq 0. \quad \text{. . . . . (14)}$$

The obvious symmetry between conditions (13) and (14) suggests that they could have been obtained from more general considerations.

15.5. *Stability of Type* (2).

Let us now discuss the second type of stability. Let us assign some value to ν (ν small). When (μ, ν) lies below the curve $\left[\frac{d^2}{d\phi^2}(C+C')\right]_{\phi=0} = 0$, the curve $C + C'$), is convex upwards in the neighbourhood of $\phi = 0$; $(C + C')$ decreases till it attains a minimum, then it increases—cuts the axis of ϕ and remains positive for all succeeding values of ϕ. If at the same time the point (μ, ν) lies above the curve $(B - B')_{\phi=0} = 0$, then the curve $(B - B')$ is positive when $\phi = 0$; it decreases—cuts the axis of ϕ at some point and remains negative for succeeding values of ϕ. In general the points of section with the ϕ axis for these two curves are different, but in the particular case where they coincide, the system is stable. When μ and ν are small we know that $\mu/\nu = 3 \cdot 559$, and in this case the two curves intersect on the ϕ axis at $\phi = \frac{1}{2}\pi$. This indicates that the tangent at the origin to this curve, which may be called the "Stability Curve," is $3 \cdot 559$. Another example of this type of stability is afforded by the case $\mu = 1 \cdot 00$, $\nu = 15° \, 45'$. This gives :—

$$\begin{aligned} C + C' = 1 \cdot 2505 &- 2 \cdot 1108 \cos \phi - 0 \cdot 3593 \cos 3\phi - 0 \cdot 0504 \cos 5\phi \\ &- 0 \cdot 0068 \cos 7\phi - 0 \cdot 0009 \cos 9\phi + 1 \cdot 1192 \cos 2\phi \\ &+ 0 \cdot 1373 \cos 4\phi + 0 \cdot 0184 \cos 5\phi + 0 \cdot 0025 \cos 8\phi + 0 \cdot 0003 \cos 10\phi. \end{aligned}$$

$$\begin{aligned} B - B' = 0 \cdot 1710 &+ 0 \cdot 5379 \cos \phi - 0 \cdot 0021 \cos 3\phi - 0 \cdot 0043 \cos 5\phi \\ &- 0 \cdot 0005 \cos 7\phi + 0 \cdot 3289 \cos 2\phi + 0 \cdot 0148 \cos 4\phi \\ &+ 0 \cdot 0015 \cos 6\phi + 0 \cdot 0002 \cos 8\phi. \end{aligned}$$

ϕ.	0.	$\frac{1}{6}\pi$.	$\frac{1}{4}\pi$.	$\frac{1}{3}\pi$.	$\frac{1}{2}\pi$.	$\frac{2}{3}\pi$.	$\frac{3}{4}\pi$.	$\frac{5}{6}\pi$.	π.
C + C′	0·0000	−0·0564	−0·0924	−0·0845	0·2523	1·3629	2·3609	3·5016	5·0563
B − B′	1·0474	0·7964	0·5409	0·2692	−0·1445	−0·2681	−0·2608	−0·2102	−0·0145

The two curves cut the ϕ axis at $\phi = 76°$ and the system is stable. For a greater value of ν there will be a correspondingly greater value of μ and the point of section with the ϕ axis will be nearer $\phi = 0$. The limiting case in this type of stability occurs when the point of section is at $\phi = 0$. At this point we have $(B - B')_{\phi=0} = 0$ and $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0} = 0$. That is, the "Stability Curve" must pass through the point of intersection of the curves $(B - B')_{\phi=0} = 0$ and $\left[\frac{d^2}{d\phi^2}(C + C')\right]_{\phi=0} = 0$ as in fig. 9.

15.6. Fig. 9 has been divided into régimes which indicate the form of the stream lines. Thus, the curve $u(0, -c) = 0$ is the curve giving the transition form between types 4 (*b*) and 4 (*d*) and also between 4 (*f*) and 4 (*h*). When $t = \sqrt{3}$, $d = \sqrt{3}/12$, $u(0, -c) = 0$ has two coincident roots—that is, at the point $\mu = \pi/\sqrt{3}$, $\nu = \pi/6$, the tangent to the curve $u(0, -c) = 0$ is parallel to the axis of ν. From the discussion of the case $\tau = 2 \cdot 06i$ we know that this curve passes through the points $\mu = 1 \cdot 525$, $\nu = 0 \cdot 906$ and $\mu = 1 \cdot 525$, $\nu = 0 \cdot 180$. Also by direct calculation we find that when $t = 10$, $u(0, -c)$ is zero when $d = +0$ and when $d = 2 \cdot 38$ approximately. This seems to indicate that at $\nu = 0$ and $\nu = \frac{1}{2}/\pi$ the curve $u(0, -c) = 0$ is perpendicular to the axis of ν. The following consideration shows this to be so. As $t \to \infty$, that is as $\mu \to 0$, the velocity at $(0, -c)$ becomes

$$(-\tanh 2\pi d - 2\tanh(\tfrac{1}{4}\pi t - d) + 2)\,\kappa\pi/4b. \tag{15}$$

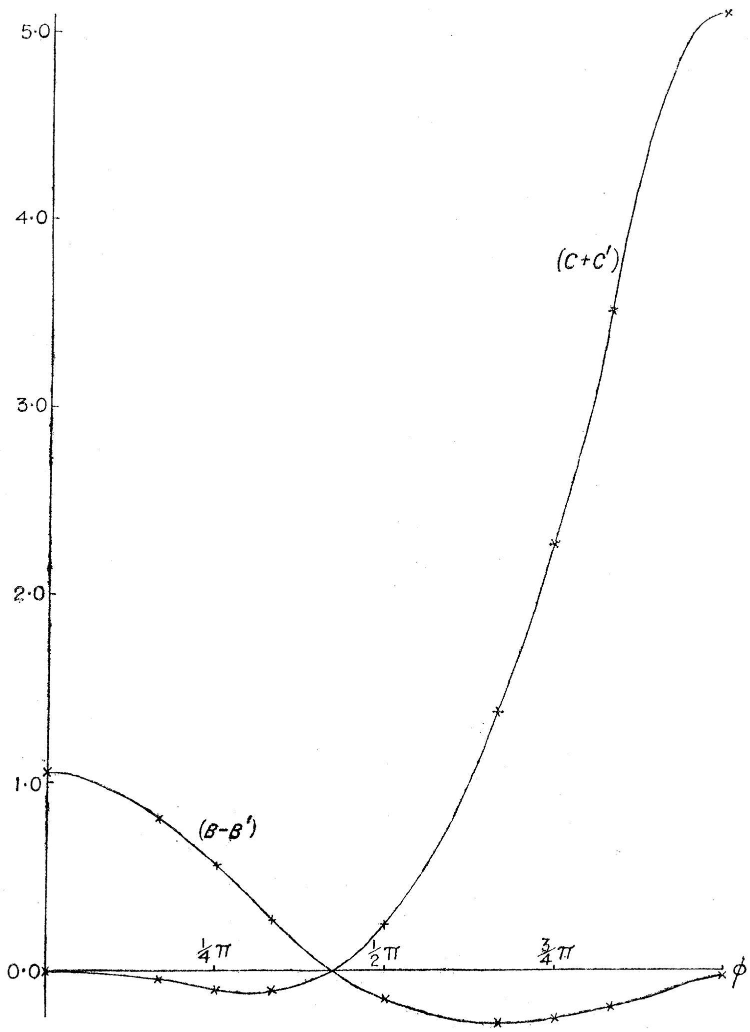

FIG. 8.

Putting $\tanh \pi d = x$, $\tanh \pi t/4 = y$, the equation $u(0, -c) = 0$ becomes

$$\frac{x}{1+x^2} + \frac{y-x}{1-xy} - 1 = 0,$$

that is

$$y = \frac{1+x+x^3}{1+x+x^2}.$$

Hence $y \to 1$ as $x \to 0$ and as $x \to 1$, that is $t \to \infty$ as $d \to 0$ and as $d \to \frac{1}{4}t$, and the approximation (15) is justified in these cases.

15.61. The limiting case between 4 (*d*) and 4 (*f*) is 4 (*e*), and this is given by the curve showing the relationship between μ and ν for which the maximum of S (0, Y) occurs at the point where S (0, Y) = 1. From the discussion of the case $\tau = 2\cdot 06i$ we know that the curve passes through the point $\mu = 1\cdot 525$, $\nu = 0\cdot 564$. The curve also passes through the point $\mu = \pi/\sqrt{3}$, $\nu = \frac{1}{6}\pi$. Also, when $t = \infty$, S (0, Y) = 1 and $\frac{\partial}{\partial y}$ S (0, Y) = 0 when $d = 0$ and Y = $-\frac{1}{4}t$. Since $2d = \nu/\mu$, the slope of the tangent

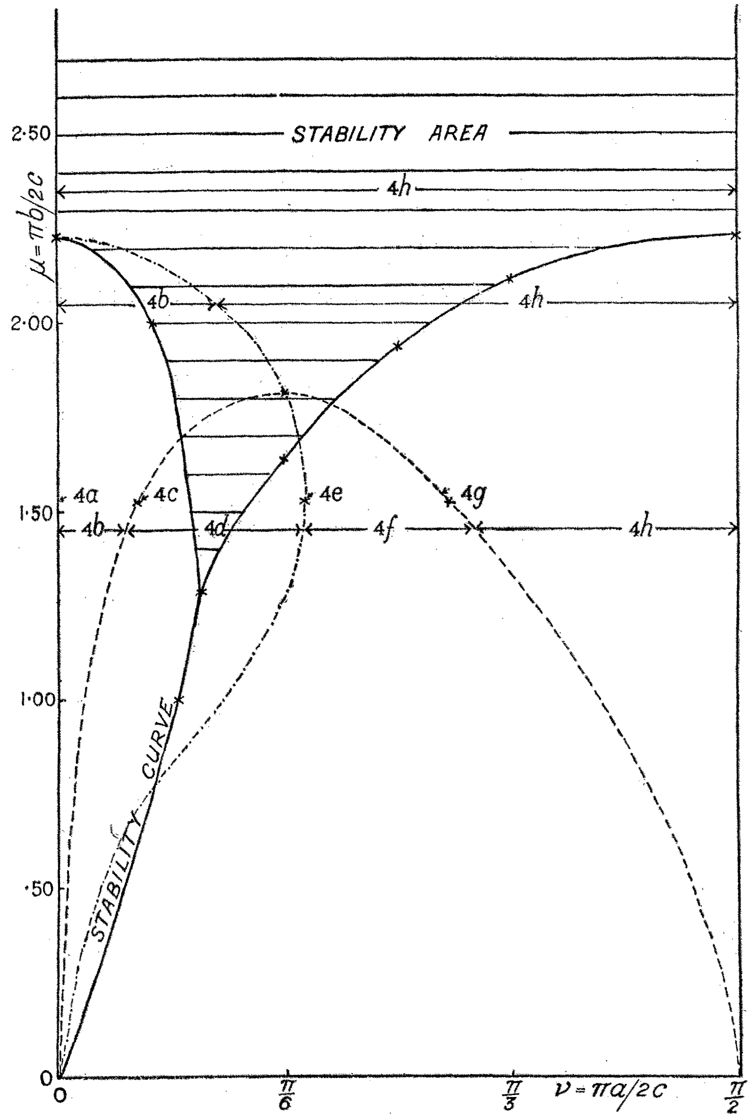

FIG. 9.

at the origin is $+\infty$, and the curve is as drawn in fig. 9. This immediately verifies the fact that when $\mu = 0$ and $\nu = 0$, the shape of the stream lines are those given by VON KÁRMÁN. When $2\cdot 229 > \mu > \pi/\sqrt{3}$, $u\,(0, -c)$ is never zero and the stream lines are either of type 4 (b) or 4 (g), with the type 6 (c) in the transition stage (see fig. 6). The curve giving the intermediate type is the curve S (X, $\frac{1}{4}t$) = 1. This curve, which passes through the points $\mu = \pi/\sqrt{3}$, $\nu = \frac{1}{6}\pi$, and $\mu = 2\cdot 229$, $\nu = 0$, has been drawn in fig. 9. When $\mu \geq 2\cdot 229$, the curves are always of the type 4 (h).

15.62. It is at once evident that when $\mu < 2\cdot 229$ there is no well-defined connection between the shape of the stream lines and the stability of the system. The stream lines are first of one type and then of another—and, in fact, when μ is slightly less than $\pi/\sqrt{3}$, the stable types include all the possible forms of stream line. It is only when $\mu \geq 2\cdot 229$ that there is a distinct connection between the stability and the shape of the stream lines—the latter are always of one type and the system is always stable.

Problem II.

16. *Determination of the* ω *Function.*

The discussion of the bounded symmetrical double row is very similar to that of the bounded unsymmetrical double row. In this system there are positive vortices at the points $(2nb, a)$ and negative vortices at the points $(2nb, -a)$; and the width of the channel is $2c$. In this case we see that $\tau\ (\equiv it) \equiv ic/b$, and using the previous notation we find that the ω function for the relative stream lines is

$$-\frac{i\kappa}{2\pi}\log \exp\left(-i\frac{2\pi Uz}{\kappa}\right)\frac{\vartheta_1(Z-id)}{\vartheta_1(Z+id)},$$

where U is the velocity of the vortices and equals $\dfrac{i\kappa}{4\pi b}\dfrac{\vartheta'_1(2id)}{\vartheta_1(2id)}$. When $t \to \infty$ we get $U \to \dfrac{\kappa}{4b}\coth \pi\dfrac{a}{b}$.

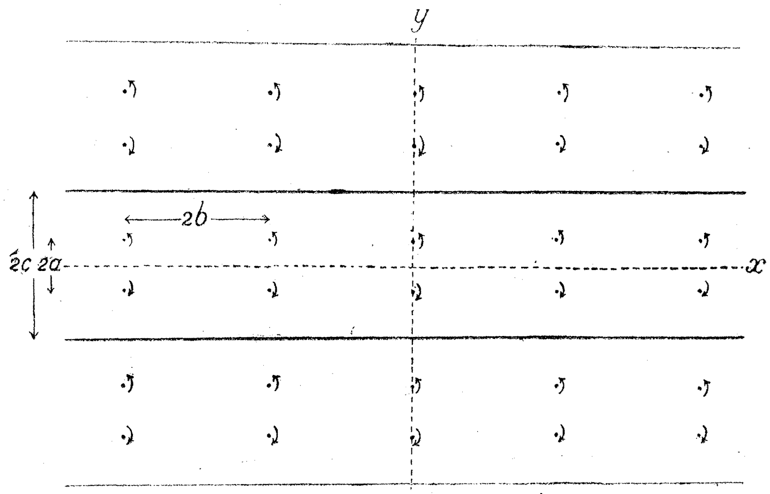

Fig. 10.

17. *Points of Zero Velocity.*

We have

$$\omega = -Uz - \frac{i\kappa}{2\pi}\log\frac{\vartheta_1(Z-id)}{\vartheta_1(Z+id)} = -Uz - \frac{i\kappa}{2\pi}\log f(Z).$$

Therefore

$$\frac{d\omega}{dz} = -\mathrm{U} - \frac{i\kappa}{4\pi b}\frac{f'(\mathrm{Z})}{f(\mathrm{Z})}.$$

As before, $f'(\mathrm{Z})/f(\mathrm{Z})$ is an elliptic function whose periods are 1 and τ. $f(\mathrm{Z})$ has one simple zero and one simple pole within a period parallelogram so that $f'(\mathrm{Z})/f(\mathrm{Z})$, and therefore $d\omega/dz$, has two simple poles within a period parallelogram. Hence $d\omega/dz$ must have one double zero or two simple zeros within this area. Therefore there must be one and only one point of zero velocity within the domain $0 \leq x \leq b$; $0 \leq y \leq c$; and it must lie on the boundaries of this domain. (It is shown later, in paragraph 18, that there cannot be a point of zero velocity on the line $x = 0$ except when $d = 0$ and $d = \frac{1}{2}t$, in which cases the velocity is zero everywhere.) When the zeros occur at the points $(b, 0)$ or (b, c) it can be shown that the zeros are of the second order. We have

$$f(\mathrm{Z})f(1-\mathrm{Z}) = 1,$$

so that

$$\frac{f'(\mathrm{Z})}{f(\mathrm{Z})} - \frac{f'(1-\mathrm{Z})}{f(1-\mathrm{Z})} = 0,$$

i.e.

$$f_1(\mathrm{Z}) - f_1(1-\mathrm{Z}) = 0,$$

and therefore

$$f'_1(\mathrm{Z}) + f'_1(1-\mathrm{Z}) = 0.$$

Putting $\mathrm{Z} = \frac{1}{2}$ we see that $f'_1(\frac{1}{2}) = 0$. Putting $\mathrm{Z} = \frac{1}{2} + \frac{1}{2}\tau$ we get

$$f'_1(\tfrac{1}{2} + \tfrac{1}{2}\tau) + f'_1(\tfrac{1}{2} - \tfrac{1}{2}\tau) = 0.$$

But $f'_1(\frac{1}{2} + \frac{1}{2}\tau) = f'_1(\frac{1}{2} - \frac{1}{2}\tau)$ since $f_1(\mathrm{Z})$ is periodic in τ. Hence $f'_1(\frac{1}{2} + \frac{1}{2}\tau) = 0$, and therefore if $d\omega/dz$ is zero at either $(b, 0)$ or (b, c) the zero is of the second order.

18. Since $t > y > -t$ we can use the following expansion

$$\frac{4b}{\kappa}\frac{d\omega}{dz} = -\frac{i}{\pi}\frac{\vartheta'_1(2id)}{\vartheta_1(2id)} - \frac{i}{\pi}\frac{\vartheta'_1(\mathrm{Z}-id)}{\vartheta_1(\mathrm{Z}-id)} + \frac{i}{\pi}\frac{\vartheta'_1(\mathrm{Z}+id)}{\vartheta_1(\mathrm{Z}+id)}$$

$$= \frac{\cosh 2d\pi(\cosh 2d\pi + \cos 2\mathrm{Z}\pi) - 2}{\sinh 2d\pi(\cosh 2d\pi - \cos 2\mathrm{Z}\pi)} + \sum_1^\infty \frac{8q^{2r}}{1-q^{2r}} \sinh 2r\pi d\,(\cosh 2r\pi d - \cos 2r\pi\mathrm{Z}).$$

When $\mathrm{Z} = i\mathrm{Y}$ or X or $\frac{1}{2} + i\mathrm{Y}$ it can easily be seen that $d\omega/dz$ is purely real, so that the velocity at any point on these three lines is parallel to the x axis. When $\mathrm{Z} = \mathrm{X} + \frac{1}{2}\tau$ we get

$$\frac{4b}{\kappa}\frac{d\omega}{dz} = -\frac{i}{\pi}\frac{\vartheta'_1(2id)}{\vartheta_1(2id)} - \frac{i}{\pi}\frac{\vartheta'_4(\mathrm{X}-id)}{\vartheta_4(\mathrm{X}-id)} + \frac{i}{\pi}\frac{\vartheta'_4(\mathrm{X}+id)}{\vartheta_4(\mathrm{X}+id)},$$

and this is also purely real since the last two terms are conjugate. When $Z = iY$ we see that $d\omega/dz$ is negative if $\frac{1}{2}t > Y > d$ and that $d\omega/dz$ is positive if $d > Y > 0$ so that there cannot be a point of zero velocity on the line $x = 0$.

19. *The Relative Stream Lines.*

19.1. The relative stream lines are given by

$$\exp\left(\frac{8\pi b \mathrm{UY}}{\kappa}\right)\left|\frac{\vartheta_1^2(Z-id)}{\vartheta_1^2(Z+id)}\right| = \exp\left(\frac{8\pi b \mathrm{UY}}{\kappa}\right)\frac{\vartheta_3^2(i\overline{Y-d})\,\vartheta_1^2(X)-\vartheta_1^2(i\overline{Y-d})\,\vartheta_3^2(X)}{\vartheta_3^2(i\overline{Y+d})\,\vartheta_1^2(X)-\vartheta_1^2(i\overline{Y+d})\,\vartheta_3^2(X)}$$

$$= \text{constant} = S(X, Y) \quad \text{say.}$$

We see immediately that the line $Y = 0$ is a stream line of every system and that it corresponds to $S(X, Y) = 1$.

19.2. When a is small the stream lines in the neighbourhood of the vortices at $(0, \pm a)$ should be similar to those of a vortex pair, since in this neighbourhood the effect of the other vortices may be neglected in comparison with the effect of the vortices at $(0, \pm a)$. Similarly when a is very nearly equal to c, the stream lines in the vicinity of the vortices at $(0, c \pm (c-a))$ should be similar to those of a vortex pair.

19.3. We have also

$$(\omega)_d = -\frac{i\kappa}{4\pi b}\frac{\vartheta'_1(2id)}{\vartheta_1(2id)}z - \frac{i\kappa}{2\pi}\log\frac{\vartheta_1(Z-id)}{\vartheta_1(Z+id)},$$

$$(\omega)_{\frac{1}{2}t-d} = \frac{i\kappa}{4\pi b}\frac{\vartheta'_1(2id)}{\vartheta_1(2id)}z - \frac{i\kappa}{2\pi}\log\frac{\vartheta_4(Z+id)}{\vartheta_4(Z-id)},$$

and we get

$$\left[\frac{\partial}{\partial z}(\omega)_d\right]_z = -\left[\frac{\partial}{\partial z}(\omega)_{\frac{1}{2}t-d}\right]_{z+\frac{1}{2}\tau}.$$

This means that if we have two systems given by the parameters d and $(\frac{1}{2}t-d)$, and if we compare the velocity at a point z of the first system with the velocity at a point $z+\frac{1}{2}\tau$ of the second system, we find that the velocities are equal in magnitude but opposite in direction. This is obvious from general considerations, for the systems d and $(\frac{1}{2}t-d)$ are identical in every respect except that one system has been pushed up through half a period parallel to the y axis. Therefore if we have the sequence of diagrams which represents the gradual change in the shape of the stream lines as d increases from 0 to $\frac{1}{4}t$, then by pushing the diagrams up by half a period and reversing the order of the diagrams we get the sequence from $\frac{1}{4}t$ to $\frac{1}{2}t$. By symmetry therefore one might expect that when $d = \frac{1}{4}t$, there should be a point of zero velocity at $(\frac{1}{2}+\frac{1}{4}\tau)$. This is correct and can easily be verified.

19.4. In view of these considerations there is one and only one sequence of diagrams for all values of τ.

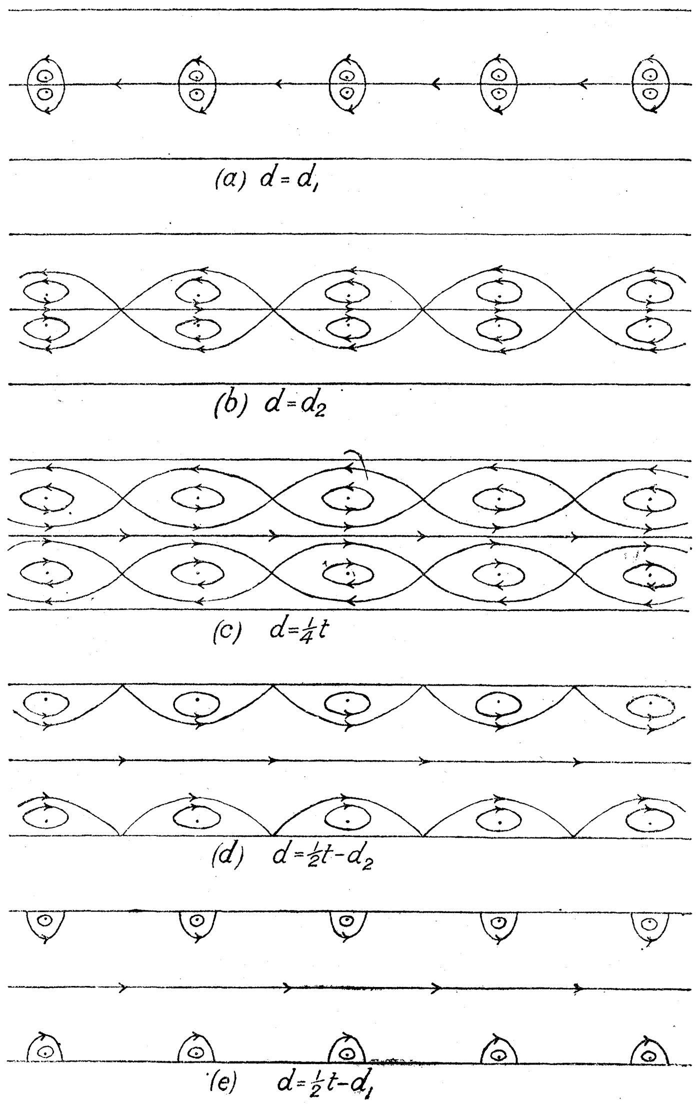

FIG. 11.

20. *Stability.*

The following discussion is sufficient to show that all these systems are unstable. The equations dealing with the disturbed motions of the vortices can be derived by following out arguments identical with those used in connection with the first problem, but they can be obtained immediately by putting $2q$ wherever $(2q-1)$ appeared in the previous work, the summation with respect to q being over all integral values from $-\infty$ to $+\infty$ including 0, in both cases. We get

$$\mathrm{C}+\mathrm{C}'=-8\sum^{\infty}(1-\cos 2p\phi)\frac{(1-\cos 4\nu)(\cosh^2 4p\mu+\cosh 4p\mu-\cos 4\nu-1)}{(\cosh 4p\mu-1)(\cosh 4p\mu-\cos 4\nu)^2}.$$

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Excluding the trivial cases $\nu = 0$ and $\nu = \frac{1}{2}\pi$ we see that $(C + C')$ is negative for all values of ϕ, μ and ν. Also

$$B - B' = -8 \sum_1^\infty (1 + \cos 2p\phi) \frac{(1 - \cos 4\nu)(\cosh^2 4p\mu + \cosh 4p\mu - \cos 4\nu - 1)}{(\cosh 4p\mu - 1)(\cosh 4p\mu - \cos 4\nu)^2}$$
$$+ 8 \sum_1^8 \frac{\cos 2p\phi}{\sinh^2 2p\mu} - 4 \operatorname{cosec}^2 2\nu.$$

We see that at $\phi = \frac{1}{2}\pi$, $(B - B')$ is negative since

$$\left[\sum_1^\infty \frac{\cos 2p\phi}{\sinh^2 2p\mu}\right]_{\phi = \frac{1}{2}\pi} = \sum_1^\infty \frac{(-1)^p}{\sinh^2 2p\mu} = -\sum_1^\infty \left(\frac{\sinh^2 4p\mu - \sinh^2 (4p-2)\mu}{\sinh^2 4p\mu \sinh^2 (4p-2)\mu}\right) = -ve.$$

Hence the product $(C + C')(B - B')$ is positive at $\phi = \frac{1}{2}\pi$. Similarly we get that $(C - C')(B + B')$ is positive at $\phi = \frac{1}{2}\pi$. Hence all these systems are unstable.

Part II.—*The Kármán Drag Formula for a Channel of Finite Breadth.*

1. *Introduction.*

The present investigations deal with the resistance experienced by a body moving in an inviscid incompressible fluid in a direction parallel to the boundary walls which are plane. The motion is two-dimensional and we assume, as did Kármán, that behind the moving body, at a distance which is great in comparison with the dimensions of the body, the motion is periodic and that the double trail of vortices arranges itself as in an unsymmetrical double row in which the stability ratio is satisfied.

1.1. *Notation.*

At this stage let us introduce the following additional notation :—

$\bar{u}$, $\bar{v}$, = components of velocity on the boundaries of a closed portion of fluid.

λ = stability ratio = $a/b = 2d$.

$\alpha = i(a + c)$. $\beta = b - i(c - a)$, $\gamma = z + ic$.

$\alpha_1 = \alpha/2b$, $\beta_1 = \beta/2b$.

V = velocity of body in the channel.

V_1 = velocity of vortices in the rear of the body.

U = velocity of vortices in the corresponding Kármán street.

W = drag experienced by the body.

ρ = density of the fluid.

l = a linear dimension of the body in the channel.

k_D = the experimental value of the drag coefficient = $W/\rho l V^2$.

k'_D = Kármán's original value of the drag-coefficient (given by formula 2).

k''_D = the Kármán value with the correction due to the channel walls (given by formula 4B).

1.2. *Summary of Previous Results.*

For the unbounded unsymmetrical double row, the ω function is

$$-\frac{i\kappa}{2\pi}\log\frac{\sin\pi(z-ia)/2b}{\cos\pi(z+ia)/2b}.$$

The velocity of the vortices is $\frac{\kappa}{4b}\tanh\pi a/b$.

The stability condition is $\cosh^2\pi a/b=2$, that is $a/b=0\cdot281$.

The general expression for the drag experienced by the body is

$$W=\frac{\rho\kappa a}{b}(V-2U)+\frac{\rho\kappa^2}{4\pi b}. \qquad (1)$$

If the stability condition is introduced into this expression, it becomes

$$W=2\rho bV^2\left\{0\cdot795\left(\frac{U}{V}\right)-0\cdot316\left(\frac{U}{V}\right)^2\right\}=k'_D\,\rho lV^2. \qquad (2)$$

For the bounded unsymmetrical double row, the ω function is

$$-\frac{i\kappa}{2\pi}\log\frac{\vartheta_1(Z-id)\,\vartheta_3(Z-id)}{\vartheta_2(Z+id)\,\vartheta_4(Z+id)}.$$

The velocity of the vortices is

$$\frac{i\kappa}{4\pi b}\left[\frac{\vartheta'_2(2id)}{\vartheta_2(2id)}+\frac{\vartheta'_4(2id)}{\vartheta_4(2id)}\right].$$

The stability ratio is only determinate when $b/c\leq 0\cdot815$. In this range $a/c\leq 0\cdot208$. Hence the following work can only be applied to the cases where the distance between the rows is less than one-fifth (approximately) of the distance between the channel walls (see paragraph 2). When $b/c\geq 0\cdot815$ there is a gradually widening range of values of a for which a Kármán street is stable. When $b/c\geq 1\cdot419$ the system is stable for all values of a.

1.3. *Summary of Present Investigations.*

When $C\to\infty$, *i.e.*, when we have a trail of vortices in an infinite sea of liquid, the drag formula approaches the limit

$$W=\frac{\rho\kappa a}{b}(V-2U)+\frac{\rho\kappa^2}{4\pi b}. \qquad (3)$$

This agrees with the Kármán result.

In the general case, *i.e.*, when the distance between the barriers is finite the drag formula is

$$W=\frac{\rho\kappa a}{b}(V-2U(1-h))+\frac{\rho\kappa^2}{4\pi b}(1-k)+\frac{3\rho^2\kappa^2d^2}{c}, \qquad (4)$$

where

$$h = \frac{ic}{2\pi a}\left\{\frac{\vartheta'_1(\frac{1}{4}\tau + id)}{\vartheta_1(\frac{1}{4}\tau + id)} - \frac{\vartheta'_2(\frac{1}{4}\tau - id)}{\vartheta_2(\frac{1}{4}\tau - id)}\right\}$$

$$k = -\frac{c}{2\pi b}\left\{\frac{\vartheta'_1(\frac{1}{4}\tau + id)}{\vartheta_1(\frac{1}{4}\tau + id)} - \frac{\vartheta'_2(\frac{1}{4}\tau - id)}{\vartheta_2(\frac{1}{4}\tau - id)}\right\}^2$$

$$-\frac{c}{2\pi b}\left\{\frac{\vartheta''_1(\frac{1}{4}\tau + id)}{\vartheta_1(\frac{1}{4}\tau + id)} + \frac{\vartheta''_2(\frac{1}{4}\tau - id)}{\vartheta_2(\frac{1}{4}\tau - id)} - \left(\frac{\vartheta'_1(\frac{1}{4}\tau + id)}{\vartheta_1(\frac{1}{4}\tau + id)}\right)^2 - \left(\frac{\vartheta'_2(\frac{1}{4}\tau - id)}{\vartheta_2(\frac{1}{4}\tau - id)}\right)^2\right\}.$$

In the range $b/c \leq 0\cdot815$, we may put (with an accuracy of within $\frac{1}{2}$ per cent.)

$$h = \frac{c}{a}\frac{\cosh \pi a/b}{\sinh \pi c/b + \sinh \pi a/b}$$

$$k = \frac{4\pi c}{b}\frac{\cosh^2 \pi a/b}{(\sinh \pi c/b + \sinh \pi a/b)^2} - \frac{\pi c}{2b}\left(\frac{1}{\sinh^2 \pi(c+a)/2b} - \frac{1}{\cosh^2 \pi(c-a)/2b}\right).$$

In the range $b/c \leq 0\cdot815$, the stability ratio for a given parameter is unique and is given by the following approximate formula

$$\lambda = 0\cdot281 - 0\cdot006\,\mu^6,$$

that is

$$a/b = 0\cdot281 - 0\cdot090\,(b/c)^6.$$

If we put $U = \frac{\kappa}{4b}\tanh\frac{\pi a}{b}$, we get

$$k''_D = W/\rho l V^2 = \frac{4a}{l}\coth\frac{\pi a}{b}\left[\frac{U}{V} - \left\{2(1-h) - \coth\frac{\pi a}{b}\left(\frac{3a}{c} + \frac{b(1-k)}{a\pi}\right)\right\}\left(\frac{U}{V}\right)^2\right]. \quad (4A)$$

If we take account of the partial mingling of the vorticity in the turbulent region behind the body, we get, by adopting an empirical value for the extent of the annihilation of the vorticity,

$$k''_D = \frac{4a}{l}\coth\frac{\pi a}{b}\left[\frac{V_1}{V} - \left\{2(1-h) - \coth\frac{\pi a}{b}\left(\frac{3a}{c} + \frac{b(1-k)}{a\pi}\right)\right\}\left(\frac{V_1}{V}\right)^2\right]. \quad (4B)$$

This formula is obtained on the assumption that owing to the annihilation of the vorticity behind the body, the strength of the vortices in the street is $\xi\kappa$ instead of κ, where

$$\xi = 1 - \frac{2a}{c}\coth\frac{\pi a}{b} = \frac{V_1}{U} < 1.$$

Formula (4B) gives results which are in close agreement with those obtained by experiment.

2. *Observations on the Stability Condition of a Bounded* KÁRMÁN *Street.*

It appears from Part I of this paper that the stability ratio a/b is only determinate when $b/c \leq 0\cdot815$. In this range $a/c \leq 0\cdot208$, that is, the distance between the rows is less than a fifth (approximately) of the distance between the channel walls. A very good empirical formula for the stability ratio in this range is

$$\lambda = 0\cdot281 - 0\cdot006\,\mu^6,$$

that is

$$a/b = 0\cdot281 - 0\cdot090\,(b/c)^6. \quad \ldots \quad (5)$$

The formula giving λ in terms of ν can be obtained from the above formula by an iterative process but the final form is rather cumbersome and it has therefore been omitted. It is interesting to note that although the mathematical result indicates that the ratio a/b in this range should be less than $0\cdot281$, the experimental evidence seems to indicate that a/b should be greater than $0\cdot281$, even when the channel is very wide in comparison with the width of the obstacle. The experimental evidence on this point is furnished by a paper by Messrs. FAGE and JOHANSEN.* In this paper the authors indicate the difficulty of determining a/b accurately by experiment since the vorticity is spread out over considerable areas and is not concentrated in definite points as postulated by the KÁRMÁN theory. It is possible therefore that there is a certain degree of error in the experimental determination of a/b, but one thing appears to be evident—a/b is, in all but one case, greater than $0\cdot281$.

When $b/c > 0\cdot815$ there is a gradually widening range of values of a for which a KÁRMÁN street is stable. When $b/c \geq 1\cdot419$ the system is stable for all values of a. The precise meaning of the range of stability is not difficult to ascertain. A tentative explanation might be that the trail of vortices consists of superposed KÁRMÁN streets of varying stability ratios. This, however, does not explain away the difficulty, as no reason is given why, when there is a range of values of the stability ratio, only some are chosen in preference to the others. In addition, when there are several KÁRMÁN systems together, the stability question has to be discussed *ab initio*, as the following simple example shows that it is possible to superpose two stable KÁRMÁN systems and produce instability :—If the system of positive vortices at $(2mb, a)$ and negative vortices at $((2n+1)\,b, -a)$ is stable, then the system with positive vortices at $(2mb + b, a)$ and negative vortices at $((2n+1)\,b + b, -a)$ is stable also, but the combined system is the symmetrical double row which is definitely unstable. The first explanation is therefore untenable. It is evident, however, that there ought to be a distinct stability ratio. The only possible explanation of the range of stability therefore is that the stability ratio depends upon the velocity of the body. A given velocity of the body would restrict the stability ratio to lie on a particular member of a family of curves and the point given by the intersection of this curve and the KÁRMÁN range would determine the stability ratio for a given velocity and for the given dimensions of the body.

The interdependence of λ and V ought to be expected from general principles. λ can only depend on V, ρ, c, l and μ, where l is some length of the body and μ_1 is the viscosity coefficient. If λ is independent of V, then it is independent of μ_1 since these are the

* FAGE and JOHANSEN, 'Roy. Soc. Proc.,' A, vol. 116, p. 170 (1927).

only two quantities that involve the time dimension. Being independent of μ_1 it is also independent of ρ since these are the only two quantities that involve the mass dimension. Hence we are left with the fact that λ is a function of c and l, that is $\lambda = fc/l$. If this were so, there ought to be a distinct stability ratio, for λ would be completely determined by the ratio c/l. The previous work shows that this is not so, hence we must have $\lambda = f\left(\frac{\mathrm{V}l\rho}{\mu_1}, \frac{c}{l}\right)$. We see therefore that if we attempt to explain the existence of the range of stability in an inviscid fluid we are forced to introduce viscosity. The introduction of a small viscosity coefficient would probably alter the stability ratios slightly, but the general form of the results, the short stability curve and the widening range of stability (or, perhaps, just a range of stability) would probably be unaltered.

The following investigation therefore only deals with the domain in which the stability ratio is determinate—that is $b/c \leq 0 \cdot 815$. In this range $a/c \leq 0 \cdot 208$.

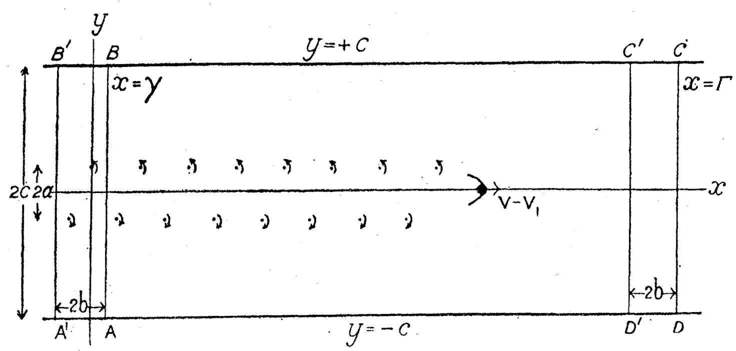

FIG. 12.

3. We consider motion relative to the moving vortices, that is, we use a set of axes that move forward with a velocity V_1. Relative to the vortices, at some distance behind the obstacle there is a uniform and periodic motion in which the components of velocity are $-\mathrm{U} + \frac{\partial \psi}{\partial y}$ and $-\frac{\partial \psi}{\partial x}$. The obstacle itself moves forward with a relative velocity $(\mathrm{V} - \mathrm{V}_1)$, and if T be the time between the shedding of consecutive vortices on the same row, we have

$$\mathrm{T}(\mathrm{V} - \mathrm{V}_1) = 2b. \quad \ldots \ldots \ldots \quad (6)$$

The origin of the axes of co-ordinate is taken to be in that part of the fluid in which the motion is that of a KÁRMÁN street. Let us consider that portion of the fluid which is bounded by the lines $y = \pm c$, $x = \gamma$, $x = \Gamma$, where $0 \leq \gamma \leq b$, and where Γ is very big and will ultimately tend to infinity. Let us denote by Δ the closed contour formed in this way. We assume that, relative to fixed axes, the liquid in the neighbourhood of $x = \Gamma$ is at rest. On the line $x = \Gamma$ the velocity relative to the moving axes is $-\mathrm{V}_1$. Hence the total flow along CD from left to right is $-2\mathrm{V}_1 c$. Along the line AB the motion is similar to that of a KÁRMÁN street and the total flow along AB from left to right is $-2Uc + \psi_B - \psi_A = -2Uc + \kappa d - (-\kappa d) = -2Uc + 2\kappa d$. There is no escape of fluid along AD and CB since they are stream lines. Since we are dealing with an incompressible fluid we must have that the flow across AB is equal to that across DC, and hence

$$V_1 = U - \kappa d/c. \qquad (7)$$

Following the method adopted by SYNGE* we arrive at the expression

$$W = D_1 - D_2$$

where

$$D_1 = \frac{\rho}{T}\int_\Delta [\phi]_0^T\, dy \quad \text{and} \quad D_2 = \tfrac{1}{2}\rho \int_\Delta \{(\bar{u}^2 - \bar{v}^2)\, dy - 2\bar{u}\bar{v}\, dx\},$$

and where $\bar{u}$, $\bar{v}$ are the components of velocity along Δ. D_1 is due to the motion of the vortices and D_2 is due to the motion relative to the vortices. In evaluating the above integrals round the contour, we assume that the lines $x = \gamma$ and $x = \Gamma$ are at great distances from the body. We assume also that it is possible to draw a fixed contour Δ_0 such that outside Δ_0 the motion is that of a KÁRMÁN street.

4. *Evaluation of* D_1.

The evaluation of D_1 will proceed on the lines laid down by SYNGE. We shall assume that as the lines $x = \gamma$ and $x = \Gamma$ tend to become infinitely distant from the body, the value of $[\phi]_0^T$ on Δ depends only on the motion of the vortex train exterior to Δ_0.

The value of ϕ at a point z due to a vortex of strength κ at z_r is

$$\phi = \frac{\kappa}{2\pi} \arg (z - z_r).$$

Considering this vortex alone, we have

$$[\phi]_0^T = \frac{\kappa}{2\pi}\,\delta \arg,$$

where δ denotes the increment of the argument in time T due to the motion of the vortex.

The value of $\int_\Delta [\phi]_0^T\, dy$ can be split up into two parts: (1) the part due to the vortices in the channel $y = \pm c$; (2) the part due to the "image vortices" which arrange themselves so as to form "image channels." Let us consider part (1). Let the point z lie in the channel $y = \pm c$ but above the vortex train. Relative to the body the vortices are moving with a velocity $V_1 - V$ and hence after time T each vortex will have moved into the position occupied by the neighbouring vortex on the same row.

* SYNGE, 'Proc. Roy. Irish Academy,' vol. 37, A. 8 (1927). (*Note*.—There is an error in Prof. SYNGE's paper—The stability condition is taken to be $\cosh^2 \pi a/b = 3$ instead of $\cosh^2 \pi a/b = 2$).

Thus, if at time t the configuration is that of ABCD, then at time $t + \mathrm{T}$, the configuration is that of A′B′C′D′. Hence we have for both the upper and lower rows of the trail that

$$\Sigma\, \delta \arg (z - z_r) = -\pi. \qquad (8)$$

Hence if z lies above the trail, then the contribution to part (1) is

$$\frac{\kappa}{2\pi}(-\pi) + \frac{-\kappa}{2\pi}(-\pi) = 0.$$

This is also true if z is below the trail. This shows incidentally that part (2) is zero, for if z is anywhere in the channel $y = \pm c$, the effect of each "image channel" is zero, and so the total effect of part (2) is zero. When z is between the rows, equation (8) still holds for the lower row, but for the upper row we have

$$\Sigma\, \delta \arg (z - z_r) = +\pi,$$

so that between the rows we have

$$[\phi]_0^{\mathrm{T}} = \frac{\kappa}{2\pi}(\pi) + \frac{-\kappa}{2\pi}(-\pi) = \kappa.$$

It is obvious also that the contribution to D_1 due to each of the lines AD, DC, CB is zero. Hence

$$\mathrm{D}_1 = \frac{\rho}{\mathrm{T}}\int_{-a}^{a} \kappa\, dy = \frac{\rho}{\mathrm{T}} 2\kappa a = \frac{\rho\kappa a}{b}(\mathrm{V} - \mathrm{U}) + \frac{2\rho\kappa^2 d^2}{\mathrm{C}}. \qquad (9)$$

5. *Evaluation of D_2.*

We have

$$\mathrm{D}_2 = \tfrac{1}{2}\rho\left[\int_{\mathrm{A}} \{(\overline{u}^2 - \overline{v}^2)\, dy - 2\overline{u}\,\overline{v}\, dx\}\right].$$

We see at once that the contribution to D_2 arising from the lines BC and DA is zero. Hence

$$\mathrm{D}_2 = \tfrac{1}{2}\rho\left[\int_{\mathrm{A}}^{\mathrm{B}} \{(\overline{u}^2 - \overline{v}^2)\, dy - 2\overline{u}\,\overline{v}\, dx\} + \int_{\mathrm{C}}^{\mathrm{D}} \{(\overline{u}^2 - \overline{v}^2)\, dy - 2\overline{u}\,\overline{v}\, dx\}\right].$$

Along CD, the components of velocity are $-\mathrm{V}_1, 0$. Hence

$$\int_{\mathrm{C}}^{\mathrm{D}} \{(\overline{u}^2 - \overline{v}^2)\, dy - 2\overline{u}\,\overline{v}\, dx\} = \int_{+\mathrm{C}}^{-\mathrm{C}} \mathrm{V}_1^2\, dy = -2c\mathrm{V}_1^2 = -2c\left[\mathrm{U}^2 - \frac{2\kappa d}{c}\mathrm{U} + \frac{\kappa^2 d^2}{c^2}\right].$$

Along AB, the components of velocity are $-\mathrm{U} + \dfrac{\partial\psi}{\partial y},\ -\dfrac{\partial\psi}{\partial x}$. Hence

$$\begin{aligned}\int_{\mathrm{A}}^{\mathrm{B}} \{(\overline{u}^2 - \overline{v}^2)\, dy - 2\overline{u}\,\overline{v}\, dx\} &= \int_{\mathrm{A}}^{\mathrm{B}} (\mathrm{U}^2 - 2\mathrm{V}u + u^2 - v^2)\, dy - 2(-\mathrm{U} + u)\, v\, dx \\ &= \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{U}^2\, dy - 2\mathrm{U}\int_{\mathrm{A}}^{\mathrm{B}} u\, dy + 2\mathrm{U}\int_{\mathrm{A}}^{\mathrm{B}} v\, dx + \int_{\mathrm{A}}^{\mathrm{B}} (u^2 - v^2)\, dy - 2uv\, dx \\ &= 2c\mathrm{U}^2 - 4\mathrm{U}\kappa d + \mathfrak{Y}\int_{\mathrm{A}}^{\mathrm{B}} \left(\frac{d\omega}{dz}\right)^2 dz,\end{aligned}$$

where $\mathfrak{I}\int$ means the imaginary part of the integral. Hence

$$D_2 = \tfrac{1}{2}\rho\left[\mathfrak{I}\int_A^B\left(\frac{d\omega}{dz}\right)^2 dz - \frac{2\kappa^2 d^2}{c}\right]. \quad \ldots \ldots \ldots \quad (10)$$

We have now to determine

$$\mathfrak{I}\int_A^B\left(\frac{d\omega}{dz}\right)^2 dz.$$

We have

$$\omega = -\frac{i\kappa}{2\pi}\log\frac{\vartheta_1(Z-id)\,\vartheta_3(Z-id)}{\vartheta_2(Z+id)\,\vartheta_4(Z+id)}.$$

Let us transfer to the point $(0, -c)$ as origin—that is, put $z + ic = \gamma$. Let us also put $i(a+c) = \alpha$ and $b - i(c-a) = \beta$. In addition let us put $\gamma/2\omega_1 = \gamma_1$, $\alpha/2\omega_1 = \alpha_1$, $\beta/2\omega_1 = \beta_1$. We get

$$\omega = -\frac{i\kappa}{2\pi}\log\frac{\vartheta_1(\gamma_1-\alpha_1)\,\vartheta_1(\gamma_1-\beta_1)}{\vartheta_1(\gamma_1+\alpha_1)\,\vartheta_1(\gamma_1+\beta_1)}.$$

Hence

$$\frac{d\omega}{dz} = -\frac{i\kappa}{4\pi b}\left[\frac{\vartheta'_1(\gamma_1-\alpha_1)}{\vartheta_1(\alpha_1-\alpha_1)} + \frac{\vartheta'_1(\gamma_1-\beta_1)}{\vartheta_1(\gamma_1-\beta_1)} - \frac{\vartheta'_1(\gamma_1+\alpha_1)}{\vartheta_1(\gamma_1+\alpha_1)} - \frac{\vartheta'_1(\gamma_1+\beta_1)}{\vartheta_1(\gamma_1+\beta_1)}\right]$$

$$= -\frac{i\kappa}{2\pi}\left[\zeta(\gamma-\alpha) + \zeta(\gamma-\beta) - \zeta(\gamma+\alpha) - \zeta(\gamma+\alpha) + \frac{2\eta_1}{\omega_1}(\alpha+\beta)\right]$$

$$= -\frac{i\kappa}{2\pi}\left\{\frac{\wp'(\alpha)}{\wp(\gamma)-\wp(\alpha)} + \frac{\wp'(\beta)}{\wp(\gamma)-\wp(\beta)} - \frac{1}{\omega_1}\left[\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right]\right\}.$$

It should be noted that $d\omega/dz$ is an elliptic function of periods 1 and τ, and hence $\int\left(\frac{d\omega}{dz}\right)^2 dz$ can, on theoretical grounds, be obtained exactly in terms of known functions. We have

$$-\frac{4\pi^2}{\kappa^2}\left(\frac{d\omega}{dz}\right)^2 = \left\{\frac{\wp'(\alpha)}{\wp(\gamma)-\wp(\alpha)}\right\}^2 + \left\{\frac{\wp'(\beta)}{\wp(\gamma)-\wp(\beta)}\right\}^2 + \frac{1}{\omega_1^2}\left\{\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right\}^2$$

$$+ \frac{\wp'(\alpha)}{\wp(\gamma)-\wp(\alpha)}\left\{\frac{2\wp'(\beta)}{\wp(\alpha)-\wp(\beta)} - \frac{2}{\omega_1}\left[\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right]\right\}$$

$$+ \frac{\wp'(\beta)}{\wp(\gamma)-\wp(\beta)}\left\{\frac{2\wp'(\alpha)}{\wp(\beta)-\wp(\alpha)} - \frac{2}{\omega_1}\left[\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta_1'(\beta_1)}{\vartheta_1(\beta_1)}\right]\right\}$$

$$= \left\{\frac{\wp'(\alpha)}{\wp(\gamma)-\wp(\alpha)}\right\}^2 + \left\{\frac{\wp'(\beta)}{\wp(\gamma)-\wp(\beta)}\right\}^2 + \frac{1}{\omega_1^2}\left\{\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right\}^2$$

$$+ \frac{ip\,\wp'(\alpha)}{\wp(\gamma)-\wp(\alpha)} + \frac{iq\,\wp'(\beta)}{\wp(\gamma)-\wp(\beta)}$$

where

$$ip = \frac{1}{\omega_1}\left\{\frac{\vartheta'_1(\alpha_1-\beta_1)}{\vartheta_1(\alpha_1-\beta_1)} - \frac{\vartheta'_1(\alpha_1+\beta_1)}{\vartheta_1(\alpha_1+\beta_1)} - \frac{2\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)}\right\} = \text{pure imaginary},$$

and

$$iq = \frac{1}{\omega_1}\left\{\frac{\vartheta'_1(\beta_1-\alpha_1)}{\vartheta_1(\beta_1-\alpha_1)} - \frac{\vartheta'_1(\beta_1+\alpha_1)}{\vartheta_1(\beta_1+\alpha_1)} - \frac{2\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right\} = \text{pure imaginary.}$$

We notice that the limits for the new variable are γ and $\gamma+\omega_3$, where γ is real and ω_3 imaginary. We have

$$\begin{aligned} \mathrm{I}_\alpha &= \int_\gamma^{\gamma+\omega_3}\left\{\frac{\wp'(\alpha)}{\wp(\gamma)-\wp(\alpha)}\right\}^2 d\gamma \\ &= \int_\gamma^{\gamma+\omega_3}\left\{\frac{\wp''(\alpha)}{\wp'(\alpha)}[\zeta(\gamma+\alpha)-\zeta(\gamma-\alpha)-2\zeta(\alpha)]-[\zeta'(\gamma+\alpha)+\zeta'(\gamma-\alpha)-2\zeta'(\alpha)]\right\}d\gamma \\ &= \left\{\frac{\wp''(\alpha)}{\wp'(\alpha)}\left[\log\frac{\sigma(\gamma+\alpha)}{\sigma(\gamma-\alpha)}-2\gamma\zeta(\alpha)\right]-[\zeta(\gamma+\alpha)+\zeta(\gamma-\alpha)-2\gamma\zeta'(\alpha)]\right\}_\gamma^{\gamma+\omega_3} \\ &= \mathfrak{R}_\alpha + i\mathfrak{J}_\alpha \end{aligned}$$

where $\mathfrak{R}_\alpha$ = real part of I_α

$$= \frac{\wp''(\alpha)}{\wp(\alpha)}\left[\log\frac{\vartheta_4(\gamma_1+\alpha_1)\,\vartheta_1(\gamma_1-\alpha_1)}{\vartheta_1(\gamma_1+\alpha_1)\,\vartheta_4(\gamma_1-\alpha_1)}\right] - \tfrac{1}{2}\left[\frac{\wp'(\gamma+\alpha)}{\wp(\gamma+\alpha)-l_3}+\frac{\wp'(\gamma-\alpha)}{\wp(\gamma-\alpha)-l_3}\right],$$

and

$$\begin{aligned} \mathfrak{J}_\alpha &= \text{imaginary part of } \mathrm{I}_\alpha \\ &= -i\frac{\wp''(\alpha)}{\wp'(\alpha)}[2\eta_3\alpha - 2\omega_3\zeta(\alpha)] + i[2\eta_3 - 2\omega_3\zeta'(\alpha)] \\ &= \frac{i\omega_3}{\omega_1^2}\left[\frac{\vartheta'_1(2\alpha_1)}{\vartheta_1(2\alpha_1)} - \frac{2\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)}\right]\left[\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)}+\frac{\pi i\alpha}{\omega_3}\right] \\ &\qquad - \frac{i\omega_3}{\omega_1}\left[\frac{\pi i}{\omega_3}+\frac{1}{2\omega_3}\left\{\frac{\vartheta''_1(\alpha_1)}{\vartheta_1(\alpha_1)} - \left(\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)}\right)^2\right\}\right]. \end{aligned}$$

Similarly if

$$\mathrm{I}_\beta = \int_\gamma^{\gamma+\omega_3}\left\{\frac{\wp'(\beta)}{\wp(\gamma)-\wp(\beta)}\right\}^2 d\gamma = \mathfrak{R}_\beta + i\,\mathfrak{J}_\beta,$$

we have

$$\begin{aligned} \mathfrak{R}_\beta &= \frac{\wp''(\beta)}{\wp(\beta)}\left[\log\frac{\vartheta_4(\gamma_1+\beta_1)\,\vartheta_1(\gamma_1-\beta_1)}{\vartheta_1(\gamma_1+\beta_1)\,\vartheta_4(\gamma_1-\beta_1)}\right] - \tfrac{1}{2}\left[\frac{\wp'(\gamma+\beta)}{\wp(\gamma+\beta)-l_3}+\frac{\wp'(\gamma-\beta)}{\wp(\gamma-\beta)-l_3}\right] \\ &\qquad - \frac{\pi i}{\omega_1}\left[\frac{\vartheta'_1(2\beta_1)}{\vartheta_1(2\beta_1)} - \frac{2\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right], \end{aligned}$$

$$\begin{aligned} \mathfrak{J}_\beta &= \frac{i\omega_3}{\omega_1^2}\left[\frac{\vartheta'_1(2\beta_1)}{\vartheta_1(2\beta_1)} - \frac{2\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right]\left[\frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}+\frac{\pi i(\beta-\omega_1)}{\omega_3}\right] \\ &\qquad - \frac{i\omega_3}{\omega_1}\left[\frac{\pi i}{\omega_3}+\frac{1}{2\omega_1}\left\{\frac{\vartheta''_1(\beta_1)}{\vartheta_1(\beta_1)} - \left(\frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right)^2\right\}\right]. \end{aligned}$$

Similarly if

$$\mathrm{I}'_\alpha = \int_\gamma^{\gamma+\omega_3}\frac{\wp'(\beta)}{\wp(\gamma)-\wp(\alpha)}\,d\gamma,$$

we get

$$\mathfrak{R}'_{\alpha} = \frac{\omega_3}{\omega_1}\left[\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\pi i \alpha}{\omega_3}\right], \quad \mathfrak{I}'_{\alpha} = i \log \frac{\vartheta_4(\gamma_1+\alpha_1)\,\vartheta_1(\gamma_1-\alpha_1)}{\vartheta_1(\gamma_1+\alpha_1)\,\vartheta_4(\gamma_1-\alpha_1)},$$

$$\mathfrak{R}'_{\beta} = \frac{\omega_3}{\omega_1}\left[\frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)} + \frac{\pi i (\beta-\omega_1)}{\omega_3}\right], \quad \mathfrak{I}'_{\beta} = \pi + i \log \frac{\vartheta_4(\gamma_1+\beta_1)\,\vartheta_1(\gamma_1-\beta_1)}{\vartheta_1(\gamma_1+\beta_1)\,\vartheta_4(\gamma_1-\beta_1)}.$$

We see therefore that

$$-\frac{4\pi^2}{\kappa^2}\,\mathfrak{I}\int_A^B\left(\frac{d\omega}{dz}\right)^2 dz = \mathfrak{I}_{\alpha} + \mathfrak{I}_{\beta} + p\,\mathfrak{R}'_{\alpha} + q\,\mathfrak{R}'_{\beta} - \frac{i\omega_3}{\omega_1^2}\left\{\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right\}^2$$

$$= \frac{i\omega_3}{\omega_1^2}\left\{\frac{\vartheta'_2(2id)}{\vartheta_2(2id)} + \frac{\vartheta'_4(2id)}{\vartheta_4(2id)}\right\}\left\{\frac{\pi i(\alpha+\beta-\omega_1)}{\omega_3} + \frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right\}$$

$$+\frac{2\pi}{\omega_1} - \frac{i\omega_3}{\omega_1^2}\left\{\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right\}^2$$

$$-\frac{i\omega_3}{2\omega_1^2}\left\{\frac{\vartheta_1''(\alpha_1)}{\vartheta_1(\alpha_1)} + \frac{\vartheta_1''(\beta_1)}{\vartheta_1(\beta_1)} - \left(\frac{\vartheta'_1(\alpha_1)}{\vartheta_1(\alpha_1)}\right)^2 - \left(\frac{\vartheta'_1(\beta_1)}{\vartheta_1(\beta_1)}\right)^2\right\}.$$

If in this expression we put $\omega_3 = 2ic$, $\omega_1 = b$, and

$$\mathrm{U} = \frac{i\kappa}{4\pi b}\left[\frac{\vartheta'_2(2id)}{\vartheta_2(2id)} + \frac{\vartheta'_4(2id)}{\vartheta_4(2id)}\right],$$

we get

$$\mathrm{D}_2 = \rho\left[\frac{\kappa a}{b}\,\mathrm{U}(1-2h) - \frac{\kappa^2}{4\pi b}(1-k) - \frac{\kappa^2 d^2}{c}\right],$$

where

$$h = \frac{ic}{2\pi a}\left\{\frac{\vartheta'_1(\frac{1}{4}\tau+id)}{\vartheta_1(\frac{1}{4}\tau+id)} - \frac{\vartheta'_2(\frac{1}{4}\tau-id)}{\vartheta_2(\frac{1}{4}\tau-id)}\right\}$$

and

$$k = -\frac{c}{\pi b}\left\{\frac{\vartheta'_1(\frac{1}{4}\tau+id)}{\vartheta_1(\frac{1}{4}\tau+id)} - \frac{\vartheta'_2(\frac{1}{4}\tau-id)}{\vartheta_2(\frac{1}{4}\tau-id)}\right\}^2$$

$$-\frac{c}{2\pi b}\left\{\frac{\vartheta''_1(\frac{1}{4}\tau+id)}{\vartheta_1(\frac{1}{4}\tau+id)} + \frac{\vartheta''_2(\frac{1}{4}\tau-id)}{\vartheta_2(\frac{1}{4}\tau-id)} - \left(\frac{\vartheta'_1(\frac{1}{4}\tau+id)}{\vartheta_1(\frac{1}{4}\tau+id)}\right)^2 - \left(\frac{\vartheta'_1(\frac{1}{4}\tau-id)}{\vartheta_1(\frac{1}{4}\tau-id)}\right)^2\right\}.$$

This gives

$$\mathrm{W} = \frac{\rho\kappa a}{b}(\mathrm{V} - 2\mathrm{U}(1-h)) + \frac{\rho\kappa^2}{4\pi b}(1-k) + \frac{3\rho\kappa^2 d^2}{c}. \quad \ldots \ldots \quad (11)$$

As $c \to \infty$ we have

$$\mathrm{W} \to \frac{\rho\kappa a}{b}(\mathrm{V} - 2\mathrm{U}) + \frac{\rho\kappa^2}{4\pi b}, \quad \ldots \ldots \ldots \ldots \quad (12)$$

since h, k and $1/c$ each tend to zero as $c \to \infty$. We see, therefore, that the case of the Kármán street in an infinite sea of liquid may be taken as the limiting case of a trail of vortices in a channel of finite breadth.

6. *The Drag in the Range* $b/c \leq 0 \cdot 815$.

In the range $b/c \leq 0 \cdot 815$, $q \leq \exp\left(-\frac{2\pi}{0 \cdot 815}\right) = 0 \cdot 00045$ and if q is neglected, we are working to an accuracy of within $\frac{1}{2}$ per cent. We may, therefore, put

$$\left.\begin{aligned} U &= \frac{\kappa}{4b} \tanh \frac{\pi a}{b}, \\ h &= \frac{c}{a} \frac{\cosh \pi a/b}{\sinh \pi c/b + \sinh \pi a/b} \\ k &= \frac{4\pi c}{b} \frac{\cosh^2 \pi a/b}{(\sinh \pi c/b + \sinh \pi a/b)^2} - \frac{\pi c}{2b}\left[\frac{1}{\sinh^2 \pi (c+a)/2b} - \frac{1}{\cosh^2 \pi (c-a)/2b}\right] \end{aligned}\right\} . \quad (13)$$

This gives

$$W = 4a \coth \frac{\pi a}{b}\left[\left(\frac{U}{V}\right) - \left\{2(1-h) - \coth \frac{\pi a}{b}\left(\frac{3a}{c} + \frac{b(1-k)}{a\pi}\right)\right\}\left(\frac{U}{V}\right)^2\right] \rho V^2$$
$$= k''_D \, \rho l V^2. \quad . \quad . \quad . \quad . \quad (14)$$

7. *Comparison with Experimental Results.*

This formula can be tested owing to the existence of some experimental data. The experimental results required for this work are given by Messrs. FAGE and JOHANSEN in Table X of the paper* mentioned previously, and they are inserted, with appropriate alterations in notation, in the first four columns of the following table. In these experiments, a flat plate of width $5 \cdot 95$ inches was used as the obstacle and the walls of the channel were 84 inches apart. The plate was put at various angles to the incident stream of air. The angles of incidence (i) are given in the first column. Instead of tabulating W directly, we tabulate the non-dimensional drag coefficient $k_D = W/\rho l V^2$ where l is the length of the obstacle and V is the velocity of the incident stream. All the cases considered fall within the range in which the mathematical stability ratio is determinate. In all the following cases the stability ratio a/b, to three places of decimals, is found to be $0 \cdot 281$. We also find that k can be neglected. By virtue of equations (7) and (13) we have

$$\frac{V_1}{V} = \left(1 - \frac{2a}{c} \coth \frac{\pi a}{b}\right) \frac{U}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If we insert the value of (U/V) given by equation (15) into equation (14), we find that the calculated values of the drag coefficient are very much greater than the experimental values. This could have been expected for we have assumed that all the vorticity generated at the sides of the body travels downstream in the form of vortices. This is not so in practice. Some of the vorticity is annulled by the partial mingling of

* FAGE and JOHANSEN, *loc. cit.*

the two streams of vorticity in the turbulent region behind the body. This view was expressed by PRANDTL* in an appendix to a paper by HEISENBERG, and is confirmed by the experiments of FAGE and JOHANSEN (*loc. cit.*). This view was adopted by GLAUERT† and he obtained the result‡

$$k''_D = k'_D + 32\frac{a^2}{lc}\left(\frac{U}{V}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If the mingling of the streams of vorticity is to be accounted for in the determination of k''_D, we must put $\xi\kappa$ where previously we put κ, subject to the condition that $\xi < 1$. In the case of the flat plate as obstacle, if we adopt the purely empirical value

$$\xi = 1 - \frac{2a}{c}\coth\frac{\pi a}{b} = \frac{V_1}{U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

we find that

$$k''_D = \frac{4a}{l}\coth\frac{\pi a}{b}\left[\frac{V_1}{V} - \left\{2(1-h) - \coth\frac{\pi a}{b}\left(\frac{3a}{c} + \frac{b(1-k)}{a\pi}\right)\right\}\left(\frac{V_1}{V}\right)^2\right]. \quad . \quad (18)$$

This gives results which are in close agreement with experiment and are slightly better than the values given by GLAUERT's determination.

Wind tunnel experiments.				U/V.	2h.	k_D'.	k_D''.		
i.	2b/l.	V_1/V.	k_D.				Equation (4A).	Equation (4B).	GLAUERT'S value.
90	5·25	0·235	1·065	0·334	0·012	0·889	1·420	0·993	1·166
70	4·85	0·245	0·975	0·337	0·006	0·853	1·307	0·948	1·093
60	4·44	0·240	0·850	0·320	0·003	0·766	1·122	0·843	0·948
50	4·08	0·210	0·690	0·273	0·001	0·625	0·872	0·674	0·737
40	3·55	0·185	0·505	0·231	—	0·484	0·637	0·514	0·545
30	2·76	0·160	0·325	0·198	—	0·329	0·402	0·342	0·354

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* 'Phys. Z.,' vol. 23, p. 363 (1922).
† 'Roy. Soc. Proc.,' A, vol. 120, p. 34 (1928).
‡ GLAUERT, *loc. cit.*, equation 29.